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MATHEMATICAL METHODS IN ELECTROMAGNETIC THEORY

Generalization of the Power Conservation Law for Scalar Mode-Diffraction Problems

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ABSTRACT: A complete form of the generalized power conservation law for the problem of mode diffraction in the lossless multi-port H - (E -) plane waveguide transformer has been obtained. The generalization has been accomplished by the use of the second Lorentz lemma in addition to the theorem of complex power. Some new equivalent forms of this law have been used to establish the fundamental properties of the scattering matrices. Operator matrix forms of the energy-conservation statement in the case of abrupt discontinuity are presented. The obtained results are intended for application in the analytic-numerical methods based on the modal analysis.

INTRODUCTION

Generalization of analytical forms of the power conservation law (PCL) in the stationary mode-diffraction theory is stimulated by the continuous development of the used mathematical apparatus from the techniques of the circuit theory (see, e.g., [1]) to the methods of the operator theory in the Hilbert [2,3] and Pontrjagin space [4].

The “generalization” of the PCL formulated for the propagating wave of the single-mode range is usually understood as extension of its applicability either to the multiwave operating regime of a device or to a finite number of “accessible” evanescent modes (as, e.g., in [5,6]). This standpoint is widely known [6,7] and may be attributed to a stereotyped transfer of such object of the transmission line theory as the *unitary* scattering matrix to the mode-diffraction theory.

In this paper we put a somewhat different sense into the term “generalized power conservation law (GPCL)”. In all papers and books, which are available for the authors, the familiar forms of this law are derived solely from the complex power theorem. Meanwhile, the second Lorentz theorem for the complex power flux from two independent sources is valid in phasor domain as well [8]. Disregarding this basic theorem for the considered problems results in

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that each correct analytical form of the PCL will be merely a certain component of the general relation.

The present paper is aimed at deriving a maximum form of the GPCL (i.e., as general form as possible) in terms of the generalized scattering matrix (GSM) for the problem of mode diffraction in lossless H - (E -) plane multi-port waveguide transformer of arbitrary geometry (including the problem of wave scattering by abrupt discontinuity).

We will pursue the following logic pattern when presenting the obtained results. At first, we introduce a space with an indefinite metric, namely, the Pontrjagin space Π, so that the all matrix operators (or infinite matrices) under consideration are defined in Π. Next, the operators of wave reflection and transmission are formed into finite operator matrices. These basic operators are used to define the GSM of the waveguide transformer and the corresponding characteristic operator. Then, the sought-for GPCL is derived from the second Lorentz lemma and the complex power theorem having regard to the principle of superposition. Next, we prove the relevant operator identities containing the used canonical symmetry of the Pontrjagin space. Application of the proven identities to the GSM and to the reflection operator (for the problem of mode diffraction by abrupt discontinuity) allows us to find new operator forms of the GPCL.

PROBLEM FORMULATION

Consider the problem of diffraction of LM - (LE -) modes in an H - (E -) plane N -port $\left(N = 1, 2, ...\right)$ waveguide transformer of the standard structure [1,3,7]. The region of wave interaction and the regular waveguides transferring the energy are homogeneous along the Cartesian coordinate axis oriented perpendicular to the H - (E -) plane. The device is filled with a homogeneous lossless medium and all metallic walls are assumed to be perfect electric conductors, while the waveguide arms are terminated in matching loads. The volume V of the domain of field determination enclosed by the metal walls of the wave interaction region and by reference planes S_n, with $n = \overline{1, N}$, which are placed in the regular waveguides, are supposed to be free of field sources. The limit $V \to 0$ is treated as the degeneration of the waveguide transformer into an abrupt discontinuity. The convention of time dependence is $\exp\left(i\omega t\right)$, and $k = \omega\sqrt{\varepsilon\mu}$ with $\operatorname{Im} k = 0$ is the wavenumber.

The electromagnetic field inside this waveguide transformer is completely determined by a scalar phasor U (a component of the field or Hertz vector along the above-mentioned Cartesian coordinate axis), which satisfies the Helmholtz equation and homogeneous boundary conditions on the metal walls.

Let the mode composition of the incident field at each of the N ports be described by an infinite row vector of complex-valued amplitudes $\mathbf{b}_n \in \ell_2$, with $n = \overline{1, N}$. Then the vector of amplitudes of the specified sources $\mathbf{b} \equiv \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_N\}$ belongs to the Hilbert space $h \equiv (\ell_2)^N$.

Let M_n be the number of modes above cutoff in the port n. Now, we will introduce an orthoprojector associated with this port, viz.

$$\mathbf{P}_{M_n} \equiv \left\{ P_{mq}^{M_n} = \sum_{p=1}^{M_n} \delta_{mp} \delta_{pq};\ m, q = \overline{1, \infty} \right\},$$

where δ_{mn} is the Kronecker delta, and then use it to create an operator matrix of projection on all propagating waves existing in the N ports

$$\mathbf{P} \equiv diag\left(\mathbf{P}_{M_1}, \mathbf{P}_{M_2}, ..., \mathbf{P}_{M_N}\right). \tag{1}$$

According to this definition, $\mathbf{P}$ represents the operator of finite rank $\chi = Tr(\mathbf{P}) = \sum_{n=1}^{N} M_n$ (throughout what follows we will assume $\chi \neq 0$). Next, the orthoprojector on all evanescent modes is $\mathbf{Q} = \mathbf{I} - \mathbf{P}$, where $\mathbf{I}$ stands for the identity operator. The existence of two mutually orthogonal subspaces of vectors of amplitudes of the propagating waves and evanescent modes allows us to introduce the Pontrjagin space $\Pi_\chi \equiv \mathbf{P}h \bigcup \mathbf{Q}h$ [4]. The canonical symmetry of this space is given by the formulae [9]

$$\mathbf{J} \equiv \mathbf{Q} - \mathbf{P} = \mathbf{I} - 2\mathbf{P} \quad \Rightarrow \quad \mathbf{J} = \mathbf{J}^{-1} = \mathbf{J}^{\dagger}, \tag{2}$$

where the dagger "$\dagger$" denotes the Hermitian conjugation.

We will describe the n-th port by the matrix operator of mode reflection ${}^{n}\mathbf{R} : \ell_2 \to \ell_2$ and by the unitary operator $\mathbf{U}_{M_n} = \mathbf{Q}_{M_n} - i\,\mathbf{P}_{M_n}$, which will be referred to as the *portal operator*. The matrix operator of mode transmission from waveguide p into waveguide q will be denoted as ${}^{pq}\mathbf{T} : \ell_2 \to \ell_2$. Note that we use the standardized operators ${}^{n}\mathbf{R}$ and ${}^{pq}\mathbf{T}$ (see, for example, [10,11]).

Then the waveguide transformer under consideration can be described using the reflection and the portal operator matrices $\Pi_\chi \to \Pi_\chi$

$$\mathbf{S}_R \equiv diag\left({}^1\mathbf{R}, {}^2\mathbf{R}, ..., {}^N\mathbf{R}\right), \tag{3}$$

$$\mathbf{U} \equiv diag\left(\mathbf{U}_{M_1}, \mathbf{U}_{M_2}, ..., \mathbf{U}_{M_N}\right), \tag{4}$$

respectively, and the transmission operator matrix $\Pi_\chi \to \Pi_\chi$

$$\mathbf{S}_T = \begin{bmatrix} 0 & {}^{12}\mathbf{T} & \dots & {}^{1N}\mathbf{T} \\ {}^{21}\mathbf{T} & 0 & \dots & {}^{2N}\mathbf{T} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{N1}\mathbf{T} & {}^{N2}\mathbf{T} & \dots & 0 \end{bmatrix}. \tag{5}$$

Based on the operators Eqs. (3) through (5) let us form the generalized (operator) scattering matrix $\mathbf{S}: \Pi_\chi \to \Pi_\chi$ and the characteristic operator $\mathbf{G}: \Pi_\chi \to \Pi_\chi$ (see, for example [11-13]) after the formulae

$$\mathbf{S} \equiv \mathbf{S}_R + \mathbf{S}_T, \quad \mathbf{G} \equiv \left(\mathbf{I} + \mathbf{S}\right)\mathbf{U}\left(\mathbf{I} - \mathbf{S}^\dagger\right). \tag{6}$$

For completeness sake, we would like to note that the first (conventional) Lorentz lemma yields the familiar properties of symmetry $\mathbf{S}_R^T = \mathbf{S}_R$ and $\mathbf{S}_T^T = \mathbf{S}_T$ (see, for example [2,3]).

OPERATOR MATRIX FORM OF THE ENERGY LAW

Two fundamental relations exist for a scalar field U, which operate with the complex power flux. With the terminology of the respective energy laws of electromagnetism we have i) the complex power theorem for a single source α of the field, viz.

$$\int_S {}^\alpha U \frac{\partial\, {}^\alpha U^*}{\partial \vec{n}} dS = \int_V \left[\left|\nabla\, {}^\alpha U\right|^2 - (k^*)^2 \left|{}^\alpha U\right|^2\right] dV \tag{7}$$

and ii) the second Lorentz lemma for two independent sources α and β of the field [8]

$$\begin{cases} \int\limits_S {}^{\alpha}U \dfrac{\partial\, {}^{\beta}U^*}{\partial \vec{n}} dS = \int\limits_V \left[\nabla\, {}^{\alpha}U \cdot \nabla\, {}^{\beta}U^* - \left(k^*\right)^2\, {}^{\alpha}U\, {}^{\beta}U^* \right] dV; \\ \int\limits_S \left({}^{\alpha}U \dfrac{\partial\, {}^{\beta}U^*}{\partial \vec{n}} - {}^{\beta}U^* \dfrac{\partial\, {}^{\alpha}U}{\partial \vec{n}} \right) dS = 0. \end{cases} \tag{8}$$

Here the asterisk "$*$" is for complex conjugation. The integration in the left-hand parts of Eqs. (7) and (8) is taken over the ordered union of all the reference planes $S \equiv \bigcup_{m=1}^{N} S_m$ and $\vec{n}$ stands for the outward normal to this surface.

By substituting the modal expansion of the field at the reference planes in Eqs. (7) and (8) and taking into account the superposition principle, we arrive at the following basic relation

$$\mathbf{b}\,\mathbf{G}\,\mathbf{b}^{\dagger} = \left\| \nabla U \right\|^2_{L_2(V)} - (k^*)^2 \left\| U \right\|^2_{L_2(V)} \tag{9}$$

(details of this procedure for the case of two-port waveguide transformer can be found in [11]). In the case of no losses this formula yields the hermicity of the characteristic operator $\mathbf{G} = \mathbf{G}^{\dagger}$, which is equivalent, in view of the definitions Eqs. (2), (4) and (6), to the equalities

$$\operatorname{Im}\mathbf{G} = \mathbf{P} - \mathbf{SPS}^{\dagger} - 2\operatorname{Im}\left(\mathbf{SQ}\right) = 0\,, \tag{10}$$

$$\mathbf{G} = \operatorname{Re}\mathbf{G} = \mathbf{Q} - \mathbf{SQS}^{\dagger} + 2\operatorname{Im}\left(\mathbf{SP}\right). \tag{11}$$

The physical corollary of the formula Eq. (9) in the form of the equality Eq.(10) evidently means that the flux of the active power through the source-free and lossless closed volume is equal to zero and represents a generalized power conservation law expressed in the maximum form through an optimum number of operator matrices.

By equating the sum and difference of Eqs. (10) and (11) we arrive at new representations for the characteristic operator, viz.

$$\mathbf{G} = \mathbf{I} - \mathbf{S}\mathbf{S}^{\dagger} - 2\operatorname{Im}\left(\mathbf{SJ}\right) = \mathbf{J} - \mathbf{SJS}^{\dagger} + 2\operatorname{Im}\mathbf{S}\,. \tag{12}$$

An important role in the further transformations of the GPCL is played by the operator

$$\mathbf{V}_{\pm} = \frac{1}{\sqrt{2}}\left(\mathbf{I} \mp i\,\mathbf{SJ}\right),$$

which allows reducing the formula Eq. (12) to a compact form, viz.

$$\mathbf{I} - \mathbf{V}_{+}\mathbf{V}_{+}^{\dagger} = \mathbf{J} - \mathbf{V}_{-}\mathbf{J}\mathbf{V}_{-}^{\dagger} = \frac{1}{2}\mathbf{G}\,, \qquad (13)$$

$$\mathbf{I} - \mathbf{V}_{+}^{\dagger}\mathbf{V}_{+} = \mathbf{J} - \mathbf{V}_{-}^{\dagger}\mathbf{J}\mathbf{V}_{-} = \frac{1}{2}\mathbf{J}\mathbf{G}^{T}\mathbf{J}\,. \qquad (14)$$

The formulas Eqs. (10) through (14) are the sought-for complete forms of the GCPL for the class of problems under consideration.

SOME EQUIVALENT FORMS OF THE GPCL

The found operator forms of the GPCL Eqs. (10) to (14) can be modified in various ways to obtain its new equivalent forms revealing the fundamental properties of the scattering operators.

To that end we will use here two operator identities which can be derived in the following way. Let us form a graph space $h_{\Gamma} = h \times h$ with elements $\{\mathbf{u}, \mathbf{v}\} \in h_{\Gamma}$; $\mathbf{u}, \mathbf{v} \in h$, where the operation of summation and the scalar product are defined in the natural manner [12]. Then, following paper [9], in the graph space we will introduce a canonical symmetry

$$\mathbf{J}_{\Gamma} \equiv \begin{bmatrix} \mathbf{0} & i\,\mathbf{J} \\ -i\,\mathbf{J} & \mathbf{0} \end{bmatrix}$$

and the orthoprojectors $\mathbf{P}_{\Gamma}^{\pm} \equiv \frac{1}{2}\left(\mathbf{I} \pm \mathbf{J}_{\Gamma}\right)$. The selected form of the operator $\mathbf{J}_{\Gamma}$ provides for the desired form of the indefinite metric, viz.

$$\left[\{\mathbf{u}, \mathbf{v}\}, \{\mathbf{u}, \mathbf{v}\}\right] = 2\,\mathrm{Im}\left(\mathbf{vJu}^{\dagger}\right).$$

By definition (see, for example, [12]) the graph of the scattering operator $\mathbf{S}$ is a subspace $\Gamma_S \subseteq h_{\Gamma}$ with elements $\{\mathbf{u}, \mathbf{uS}\} \in \Gamma_S$; $\mathbf{u} \in h$ (here we have taken into account that the initial Hilbert space h has been formed by row vectors). The introduced orthoprojectors provide for the necessary canonical decomposition of the graph

$$\Gamma_S = \mathbf{P}_\Gamma^+ \Gamma_S \oplus \mathbf{P}_\Gamma^- \Gamma_S ,$$

which is determined by its elements as follows

$$\mathbf{P}_\Gamma^\pm \Gamma_S = \left\{ \left\{ \mathbf{u}\mathbf{V}_\pm , \pm i\, \mathbf{u}\mathbf{V}_\pm \mathbf{J} \right\} : \mathbf{u} \in h \right\} .$$

Making use of the Pythagorean theorem and the indefinite form for the constructed decomposition of the graph Γ_S , we can find the sought-for identities, viz.

$$\mathbf{I} + \mathbf{S}\mathbf{S}^\dagger = \mathbf{V}_+ \mathbf{V}_+^\dagger + \mathbf{V}_- \mathbf{V}_-^\dagger , \tag{15}$$

$$2\,\mathrm{Im}\left(\mathbf{S}\mathbf{J}\right) = \mathbf{V}_+ \mathbf{V}_+^\dagger - \mathbf{V}_- \mathbf{V}_-^\dagger . \tag{16}$$

By combining the relations Eqs. (15) and (16) with the GPCL Eqs. (13) and (14) we can obtain new forms of the energy conservation law. One of these is as follows

$$\frac{1}{2}\left(\mathbf{I} - \mathbf{S}\mathbf{S}^\dagger\right) = \mathbf{Q} - \mathbf{V}_- \mathbf{Q} \mathbf{V}_-^\dagger , \tag{17}$$

which describes the measure of deflection of the GSM properties from the unitarity. Namely, the greater is the number of propagating modes in the waveguide arms, the closer is (in the sense of Eq. (17)) the GSM $\mathbf{S}$ to the unitary operator, never reaching this limit. Two other found forms of the GPCL, viz.

$$\begin{cases} \mathrm{Im}\,\mathbf{S} = \mathbf{P} - \mathbf{V}_+ \mathbf{P} \mathbf{V}_+^\dagger ; \\ \mathrm{Im}\left(\mathbf{S}\mathbf{J}\right) = \mathbf{P} - \mathbf{V}_- \mathbf{P} \mathbf{V}_-^\dagger \end{cases} \tag{18}$$

determine the measure of deviation of the GSM from the self-adjoint operator. Specifically, the operators $\mathbf{S}$ and $\mathbf{S}\mathbf{J}$ are quasi-Hermitian, since their imaginary components are finite rank operators [13]. This means, in particular, that every accumulation point of the spectrum $\sigma\left(\mathbf{S}\right)$ belongs to the spectrum of the real component of the GSM, whereas every nonreal point of $\sigma\left(\mathbf{S}\right)$ represents an eigenvalue of a finite multiplicity [13].

SPECIAL CASE: AN ABRUPT DISCONTINUITY

Consider a specific case of $\mathbf{G} = 0$ or in the expanded form

$$\mathbf{Q}-\mathbf{SQS}^{\dagger}+2\,\mathrm{Im}\left(\mathbf{SP}\right)=0\,, \qquad (19)$$

which corresponds to equating to zero of the right-hand part of the formula Eq. (9). This situation is possible both for the case of an abrupt discontinuity, i.e., $V=0$ (note that when proceeding to the limit $V\to 0$ the key role is played by the “condition at the sharp edge”), and with $V\neq 0$ for certain frequencies (as, for example, in the case of resonances for the “trapped modes” in the wave interaction region). As follows from the equalities Eqs. (13) and (14), the operator $\mathbf{V}_{+}$ is a unitary one for either of these two cases, whereas the operator $\mathbf{V}_{-}$ will be J-unitary, viz.

$$\begin{cases}\mathbf{V}_{+}\mathbf{V}_{+}^{\dagger}=\mathbf{V}_{+}^{\dagger}\mathbf{V}_{+}=\mathbf{I};\\ \mathbf{V}_{-}\mathbf{J}\mathbf{V}_{-}^{\dagger}=\mathbf{V}_{-}^{\dagger}\mathbf{J}\mathbf{V}_{-}=\mathbf{J}.\end{cases} \qquad (20)$$

In addition, from Eq. (12) follows

$$\frac{1}{2}\left(\mathbf{I}-\mathbf{S}\mathbf{S}^{\dagger}\right)=\mathrm{Im}\left(\mathbf{SJ}\right), \qquad (21)$$

i.e., the expression appearing in parenthesis in the left-hand part of Eq. (21) represents a quasi-Hermitian operator. Also note that the involution property $\mathbf{S}^{2}=\mathbf{I}$ for both these cases follows from the theorem of oscillating power (see, for example, paper [11]).

Here we will consider the first special case where the surface S in the remaining integrals of the theorems Eqs. (7) and (8) degenerates into a single reference plane which corresponds to step-like variation in the cross-section or/and in the curvature of the waveguide, waveguide bifurcation, etc.

The found forms of the GPCL for $N\geq 3$ are limited to the formulas Eqs. (20) and (21) presented above. However, the circle of the obtained results is essentially wider for the practically important case of $N=2$ owing to particular properties of the scattering matrices. Namely, in this case the properties

$$\left(\mathbf{I}-\mathbf{S}_{R}\right)\left(\mathbf{I}+\mathbf{S}_{R}\right)=\mathbf{S}_{T}^{2}\ \text{ and }\ \mathbf{S}_{R}\mathbf{S}_{T}+\mathbf{S}_{T}\mathbf{S}_{R}=0$$

follows from the theorem of oscillating power and the first Lorentz lemma. With account of these relations the equality

$$\mathbf{S}_{T}\mathbf{U}\left(\mathbf{I}-\mathbf{S}_{R}^{\dagger}\right)=\left(\mathbf{I}+\mathbf{S}_{R}\right)\mathbf{U}\mathbf{S}_{T}^{\dagger}\,,$$

which arise from the second Lorentz lemma, goes over into a corollary of the complex power theorem

$$\left(\mathbf{I}+\mathbf{S}_R\right)\mathbf{U}\left(\mathbf{I}-\mathbf{S}_R^{\dagger}\right)=\mathbf{S}_T\mathbf{U}\,\mathbf{S}_T^{\dagger} \tag{22}$$

and vice versa (however, this is not correct for $N>2$). So, the formula Eq. (22) can be accepted as the basic power relation for the case $V=0$ and $N=2$.

By separating the real and imaginary components of the operator equality Eq. (22) and proceeding similar as in deriving the formula Eq. (12), we can obtain the sought-for GPCL in equivalent forms

$$\mathbf{I}-\mathbf{W}_+\mathbf{W}_+^{\dagger}=\frac{1}{2}\mathbf{S}_T\mathbf{S}_T^{\dagger}\,, \tag{23}$$

$$\mathbf{J}-\mathbf{W}_-\mathbf{J}\mathbf{W}_-^{\dagger}=\frac{1}{2}\mathbf{S}_T\mathbf{J}\mathbf{S}_T^{\dagger}\,, \tag{24}$$

where $\mathbf{W}_{\pm}\equiv\frac{1}{\sqrt{2}}\left(\mathbf{I}\mp i\,\mathbf{S}_R\mathbf{J}\right)$. The first equality yields immediately the estimates $\left\|\mathbf{S}_T\right\|\le\sqrt{2}$ and $\left\|\mathbf{W}_+\right\|\le 1$. Next, by combining the equalities Eqs. (23) and (24) with the identities

$$\mathbf{I}+\mathbf{S}_R\,\mathbf{S}_R^{\dagger}=\mathbf{W}_+\mathbf{W}_+^{\dagger}+\mathbf{W}_-\mathbf{W}_-^{\dagger}\,,$$

$$2\,\mathrm{Im}\left(\mathbf{S}_R\mathbf{J}\right)=\mathbf{W}_+\mathbf{W}_+^{\dagger}-\mathbf{W}_-\mathbf{W}_-^{\dagger}\,,$$

we can find, in particular, the following equivalent forms of the GPCL

$$\frac{1}{2}\left(\mathbf{I}-\mathbf{S}_R\,\mathbf{S}_R^{\dagger}\right)=\mathbf{P}-\mathbf{W}_-\mathbf{P}\mathbf{W}_-^{\dagger}+\frac{1}{2}\mathbf{S}_T\mathbf{Q}\mathbf{S}_T^{\dagger}=\mathbf{Q}-\mathbf{W}_-\mathbf{Q}\mathbf{W}_-^{\dagger}+\frac{1}{2}\mathbf{S}_T\mathbf{P}\mathbf{S}_T^{\dagger}, \tag{25}$$

$$\mathrm{Im}\,\mathbf{S}_R=\mathbf{P}-\mathbf{W}_+\mathbf{P}\mathbf{W}_+^{\dagger}-\frac{1}{2}\mathbf{S}_T\mathbf{P}\mathbf{S}_T^{\dagger}=-\mathbf{Q}+\mathbf{W}_+\mathbf{Q}\mathbf{W}_+^{\dagger}+\frac{1}{2}\mathbf{S}_T\mathbf{Q}\mathbf{S}_T^{\dagger}. \tag{26}$$

The latter allows concluding that the reflection operator for a two-port abrupt discontinuity is a quasi-Hermitian one.

Note that with $N=2$ the operator matrices of reflection $\mathbf{S}_R$ and transmission $\mathbf{S}_T$ are "diagonal". Therefore, the relations completely analogous to the equalities Eqs. (22) through (26) can also be constructed for individual operators ${}^{n}\mathbf{R}$ and ${}^{pq}\mathbf{T}$ (see [10,11,14,15]) composing these matrices.

DISCUSSION AND CONCLUSIONS

Thus, using jointly the complex Poynting's theorem and the second Lorentz lemma we have derived a generalized form of the power conservation law given by the operator equalities Eqs. (10) and (11). The contribution of each of these theorem to the GPCL is quite clear. The theorem of complex power is valid for the field from a single source. For this reason its direct corollary will be operator relations occupying the main diagonal of the resultant operator matrix, whereas all off-diagonal operator blocks arise from the second Lorentz theorem. Since the list of the fundamental laws of electromagnetics concerning the complex power flux is exhausted by these two theorems, it can be stated the derived forms of the GPCL Eqs. (10) through (14) are maximum complete in this sense for the circle of problems under consideration.

As can be shown, the equality Eq. (10) contains the familiar particular forms of the PCL.

Let us introduce the operator $\mathbf{S}_0 \equiv \mathbf{PSP}$, which represents a trivial completion of the classical (finite) scattering matrix to an infinite matrix by zeros. By multiplying the right- and left-hand parts of the relation Eq. (10) by the orthoprojector $\mathbf{P}$, we arrive at the widely used in practice PCL for propagating waves, viz. $\mathbf{S}_0\mathbf{S}_0^\dagger = \mathbf{S}_0^\dagger\mathbf{S}_0 = \mathbf{P}$ or in the expanded form

$$\sum_{s=(0)1}^{M_p} {}^pR_{ms}\,{}^pR_{ns}^* + \sum_{q\neq p}^{N}\sum_{s=(0)1}^{M_q} {}^{pq}T_{ms}\,{}^{pq}T_{ns}^* = \delta_{mn}; \quad m,n \le M_p, \quad p = \overline{1,N}. \quad (27)$$

This result means, in particular, that the operator $\mathbf{S}_0$ is partially isometric in h. Next, extracting the diagonal operator blocks from the operator matrix Eq. (10) we obtain

$$\sum_{s=(0)1}^{M_p} {}^pR_{ms}\,{}^pR_{ns}^* + \sum_{q\neq p}^{N}\sum_{s=(0)1}^{M_q} {}^{pq}T_{ms}\,{}^{pq}T_{ns}^* = \begin{cases} \delta_{mn}, & m,n \le M_p \\ 0, & m > M_p,\ n \le M_p \\ -2\,\mathrm{Im}\,{}^pR_{mn}, & n > M_p,\ \forall m \end{cases}, \quad (28)$$

where $p = \overline{1,N}$. This formula represents the well-known generalization of relation Eq. (27) to evanescent modes (the widely used particular cases of this equality can be found, for example, in [3]). In the case of abrupt discontinuities the formula Eq. (28) is complemented with one more similar relation. Namely, extracting the diagonal operator blocks from Eq. (19), we can obtain the equality

$$\sum_{s=M_p+1}^{\infty} {}^{p}R_{ms}\,{}^{p}R_{ns}^{*} + \sum_{q\neq p}^{N}\sum_{s=M_q+1}^{\infty} {}^{pq}T_{ms}\,{}^{pq}T_{ns}^{*} = \begin{cases} 2\,\mathrm{Im}\,{}^{p}R_{mn}, & n \le M_p,\ \forall m \\ 0, & m \le M_p,\ n > M_p \\ \delta_{mn}, & m,n > M_p \end{cases}, \quad (29)$$

where $p=\overline{1,N}$. Note that the formulae Eqs. (28) and (29), as well as their combinations can be treated as versions of the canonical “optical theorem” for waveguide systems. Then the complete formula Eq. (10) represents the generalized optical theorem for lossless and reciprocal waveguiding structures in the operator form.

Construction of the equivalent forms Eqs. (17) and (18) of the GPCL allow determining important properties of the GSM, including characteristic features of its complete spectrum. As follows from the obtained results, the GSM represents quasi-Hermitian and nonunitary operator with the rank of its nonhermicity being equal to the total number of propagating waves in the energy feeding waveguides. The whole essential spectrum of the GSM lies on the real axis, whereas each nonreal point of its spectrum represents an eigenvalue of a finite multiplicity.

In the case of degeneration of the waveguide transformer into a N -port abrupt discontinuity the complete forms of the GPCL appear as equalities Eqs. (20) and (21). For the practically important case of a two-port transformer the sought-for power relation assumes the form of the formulae Eqs. (22) through (24). Its equivalent forms Eqs. (23) and (26) allow us to estimate the norm of the transmission operator $\|\mathbf{S}_T\| \le \sqrt{2}$ and to prove that the reflection operator is a quasi-Hermitian one.

In addition, we would like to note that all the obtained formulae are invariant with respect to replacing the GSM with the operator matrix $\mathbf{S}_- \equiv \mathbf{S}_R - \mathbf{S}_T$, which could be also demanded by engineering practice.

The remarkable simplicity of the formulae derived for the GPCL in terms of operator matrices is governed evidently by the accepted idealizations whose collection though is standard for the modern electrodynamic analysis of centimeter and millimeter wave transformers.

The use of any suitable form of the power conservation law from the fourteen found formulae Eqs. (10) through (14) and (17) to (25) allows validating the law simultaneously for all the waveguide modes allowed for in the field representation through the standard procedure in the practice of computer simulations of microwave devices.

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