Telecommunications and Radio Engineering, 72(7):555-567 (2013)

MATHEMATICAL METHODS IN ELECTROMAGNETIC THEORY

GENERALIZED MODE-MATCHING TECHNIQUE IN THE THEORY OF GUIDED WAVE DIFFRACTION. PART 3: WAVE SCATTERING BY RESONANT DISCONTINUITIES*

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The problem of justification of the correctness of the matrix-operator models of the mode-matching technique as applied to the problems of resonant wave scattering by waveguide discontinuities has remained of great importance throughout the years of the intensive use of the method. Another unsolved problem is substantiation of using the truncation procedure to solving the obtained infinite matrix equations. The present paper is aimed at proving rigorously correctness of the mathematical model in the form of the operator-based Fresnel formulas for the specified class of mode diffraction, constructing projection approximations for the sought-for scattering operators and justifying their convergence. To that end a generalized mode-matching technique is used. The "generalized operator-based Fresnel formulas" are derived for the scattering operator matrices. The universality of the constructed operator model in the form of the Cayley transform is proven. It is shown that domain of correctness of this model is completely determined by the established operator properties of the generalized scattering matrix. The unconditional convergence of the projection approximations to the exact solution is proved analytically. The mode-matching technique which is widely used for solving scalar problems of waveguide mode diffraction possesses a matrix-operator nature and an adequate to this nature mathematical apparatus, specifically, the theory of operators in the Hilbert space. The suggested generalization of the mode-matching technique can be used for rigorous analysis of microwave devices.

KEY WORDS: *mode-matching technique, waveguide transformer, Cayley transformation*

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* Originally published in *Radiophysics and Electronics*, Vol. 3(17), No 4, 2012, pp. 22–28.

1. INTRODUCTION

The previous parts of the study [1,2] present fundamentals of the generalized mode-matching technique which has been used to derive the operator-based Fresnel formulas for step-like (i.e., with the inherent volume $V = 0$) discontinuity in a waveguide section. It appears that in the case of volumetric ($V \neq 0$, i.e., forming an open resonant cavity) waveguide discontinuities this approach leads as well to similar operator relations (or "generalized operator-based Fresnel formulas") however already for the generalized scattering matrix **S**.

In this part of the study we consider specific features of application of the mode-matching technique generalized through introducing scattering operators in the matrix form for analyzing the problem of mode diffraction in the general case $V \neq 0$ of hollow *H*- and *E*-plane wave transformers, including i) the technique of constructing the matrix-operator model; ii) determination of the existence condition for the sought-for solution and rigorous proof of the correctness of the developed model and iii) analytical investigation of the convergence of the reduction method approximations to the exact solution and of the behavior of the condition number of the matrix operators.

The present paper is structured as follows. First, we introduce the operator matrices necessary to completely describe the characteristics of planar wave transformers. Then in terms of the generalized scattering matrix **S** and its Cayley transform we formulate the basic energy laws which play a key role in the further analysis and are poorly presented in the available scientific literature.

Next, for the classical problem of right-angled bend of a rectangular waveguide we derive the generalized Fresnel formula for the operator **S** and prove the universality of this matrix-operator model for the whole class of the problems under consideration. Hereupon we justify the existence, uniqueness and robustness of the found solution, as well as the unconditional convergence of the projection approximations.

In the paper the basic conceptions and terminology of the two first parts of the study [1,2] are used. The necessary generalizations are presented below.

2. CLASS OF THE MODE DIFFRACTION PROBLEMS UNDER INVESTIGATIONS AND THE USED NOTATIONS

Let us consider the scalar diffraction problem for the modes LM_{m0}, $m = 1, 2, \ldots$, and LE_{m1}, $m = 0, 1, \ldots$, in a hollow *H*-plane and, respectively, *E*-plane waveguide transformer of the standard geometry with N ports, $N = 1, 2, \ldots$, [3,4]. The resonant cavity and power-feeding regular waveguides are homogeneous along the Cartesian frame axis oriented perpendicular to the *H*- (*E*-) plane. It is assumed that the domain of field determination is filled with a homogeneous lossless medium, the metal walls on the boundaries of this domain are perfectly conducting and the waveguide arms are perfectly matched with the load. The volume $V \neq 0$ enclosed by the metal walls of the wave interaction region and reference planes $\Theta_n, n=\overline{1,N}$ located in the regular waveguides is supposed to be free of field sources/sinks. The time dependence is accepted in the form $\exp(i\omega t)$, where $k=\omega\sqrt{\varepsilon\mu}$, with $\mathrm{Im}k=0$, is the wavenumber.

Let the mode composition of the incident wave in each of the N ports of the transformer be described by an infinite row-vector of complex amplitudes $\mathbf{b}_n \in \ell_2, n=\overline{1,N}$. Then the vector of amplitudes of the specified field sources $\mathbf{b}\equiv\{\mathbf{b}_1,\mathbf{b}_2,...,\mathbf{b}_N\}$ belongs to the Hilbert space $h\equiv(\ell_2)^N$. Next, let P_n modes be propagating for waveguide port n. Using the corresponding orthoprojector denoted as $\mathbf{P}_{P_n}$ we can form an operator matrix of projection on the all existing in the N ports propagating modes, viz.

$$\mathbf{P}\equiv \mathrm{diag}\left(\mathbf{P}_{P_1},\mathbf{P}_{P_2},\dots,\mathbf{P}_{P_N}\right). \tag{1}$$

By this definition $\mathbf{P}$ is an operator of a finite rank $\chi=\sum_{n=1}^{N}P_n=Tr\left(\mathbf{P}\right)$ (in what follows we will suppose that $\chi\neq 0$). The orthoprojector on the all evanescent modes is $\mathbf{Q}=\mathbf{I}-\mathbf{P}$, where $\mathbf{I}$ stands for the identity operator. (Note that the existence of two mutually orthogonal subspaces of the vectors of amplitudes of the propagating and evanescent modes makes it possible to introduce the Pontrjagin space [5] as the domain of definition of the scattering operators).

The n-th regular waveguide (or n-th port of the transformer) will be characterized by the reflection operator in the matrix form ${}^{n}\mathbf{R}$ and unitary portal operator $\mathbf{U}_{P_n}=\mathbf{Q}_{P_n}-i\mathbf{P}_{P_n}$ [5,6]. The matrix operator of the wave transmission from the port p into the port q will be denoted as ${}^{pq}\mathbf{T}$. We emphasize that the standardized operators ${}^{n}\mathbf{R}:\ell_2\to\ell_2$ and ${}^{pq}\mathbf{T}:\ell_2\to\ell_2$ are used here (see, for example, [7]).

The wave transformer under consideration can be completely described with the use of the following portal operator matrix acted in the space h

$$\mathbf{U}\equiv \mathrm{diag}\left(\mathbf{U}_{P_1},\mathbf{U}_{P_2},\ \dots\ ,\mathbf{U}_{P_N}\right) \tag{2}$$

and the generalized scattering matrix

$$\mathbf{S}=\begin{bmatrix} {}^{1}\mathbf{R} & {}^{12}\mathbf{T} & \cdots & {}^{1N}\mathbf{T} \\ {}^{21}\mathbf{T} & {}^{2}\mathbf{R} & \cdots & {}^{2N}\mathbf{T} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{N1}\mathbf{T} & {}^{N2}\mathbf{T} & \cdots & {}^{N}\mathbf{R} \end{bmatrix}. \tag{3}$$

Making use of the operators Eqs. (2) and (3) it is possible to construct the characteristic operator (see, for example, [6])

$$\mathbf{G} \equiv \left(\mathbf{I}+\mathbf{S}\right)\mathbf{U}\left(\mathbf{I}-\mathbf{S}^{\dagger}\right): h \to h, \tag{4}$$

where the dagger † stands for the Hermitian conjugation.

3. CONSERVATION LAWS AND PROPERTIES OF THE SCATTERING OPERATORS

For the diffraction problems of the class under consideration four (I-IV) fundamental laws of electromagnetism are valid. These are

I. The first Lorentz lemma which yields the symmetry property

$$\mathbf{S}^{T} = \mathbf{S} \tag{5}$$

(the superscript T means transposition);

II. The oscillating power theorem [8] which leads to the relation

$$\mathbf{b}\left(\mathbf{I}-\mathbf{S}^{2}\right)\mathbf{b}^{T} = \left(\nabla_{\|}U, \nabla_{\|}U\right)_{V} - k^{2}\left(U,U\right)_{V}, \forall \mathbf{b} \in h, \tag{6}$$

where U stands for the phasor in the volume V and $\nabla_{\|}U$ is the gradient of this phasor in the *H*- (*E*-) plane; and

III-IV. The complex power theorem and the second Lorentz lemma [9] which jointly yield the equality

$$\mathbf{b}\,\mathbf{G}\,\mathbf{b}^{\dagger} = \left\| \nabla_{\|}U \right\|^{2}_{L_2(V)} - k^{2}\left\| U \right\|^{2}_{L_2(V)}, \forall \mathbf{b} \in h. \tag{7}$$

In the formulas Eqs. (6) and (7) we have used the standard notations

$$\left(f, g\right)_{V} \equiv \int_{V} f\,g\,dV \quad \text{and} \quad \left\| f \right\|^{2}_{L_2(V)} \equiv \left(f, f^{\dagger}\right)_{V}$$

for the scalar (bilinear) product and norm of function in the space $L_2(V)$.

As a corollary of the relation Eq. (7), the energy conservation law in terms of the characteristic operator Eq. (4) takes an especially simple form, viz.

$$\operatorname{Im}\mathbf{G} \equiv \frac{1}{2i}\left(\mathbf{G}-\mathbf{G}^{\dagger}\right) = 0, \tag{8}$$

i.e., the operator $\mathbf{G}$ is a self-adjoint one. The structure of this operator Eq. (4) suggests that it is reasonable to introduce into consideration a homographic transformation of the generalized scattering matrix.

To that end let us exclude in the course of the further analysis the singular points on the frequency axis (i.e., resonance frequencies) for which, as follows from the relations Eqs. (6) and (7), we have $\mathbf{b}\left(\mathbf{I}-\mathbf{S}^2\right)\mathbf{b}^T=0$ and $\mathbf{b}\,\mathbf{G}\,\mathbf{b}^\dagger=0$ with $\forall\mathbf{b}\neq 0$. As can be easily seen, this requirement means elimination of two real numbers of infinite multiplicity from the spectrum of the generalized scattering matrix, i.e., $\pm 1\notin\sigma(\mathbf{S})$ [6].

Now we can introduce the Cayley transform of the operator $\mathbf{S}$ after the formula

$$\mathbf{W}_{\mp}(\mathbf{S})=\frac{\mathbf{I}\mp\mathbf{S}}{\mathbf{I}\pm\mathbf{S}}.\ \begin{pmatrix}H\\E\end{pmatrix} \tag{9}$$

(Note that by definition the operators $\mathbf{W}_{-}\equiv\mathbf{Y}$ and $\mathbf{W}_{+}\equiv\mathbf{Z}$ are the generalized admittance and impedance matrices, respectively.) Then the characteristic operator Eq. (4) takes the form

$$\frac{1}{4}\mathbf{G}=\left(\mathbf{W}_{\mp}+\mathbf{I}\right)^{-1}\begin{Bmatrix}\mathbf{U}\,\mathbf{W}_{\mp}^{\dagger}\\ \mathbf{W}_{\mp}\mathbf{U}\end{Bmatrix}\left(\mathbf{W}_{\mp}^{\dagger}+\mathbf{I}\right)^{-1},\ \begin{pmatrix}H\\E\end{pmatrix} \tag{10}$$

and the energy conservation law in the form of Eq. (8) is

$$\operatorname{Im}\begin{Bmatrix}\mathbf{W}_{\mp}\mathbf{U}^{-1}\\ \mathbf{W}_{\mp}\mathbf{U}\end{Bmatrix}=0.\ \begin{pmatrix}H\\E\end{pmatrix} \tag{11}$$

The latter equality can be also written in the form

$$\mathbf{W}_{\mp}=\begin{Bmatrix}\mathbf{U}\\ \mathbf{U}^{-1}\end{Bmatrix}\mathbf{W}_{\mp}^{\dagger}\begin{Bmatrix}\mathbf{U}\\ \mathbf{U}^{-1}\end{Bmatrix}.\ \begin{pmatrix}H\\E\end{pmatrix} \tag{12}$$

The meaning of this condition for the certain problem of mode diffraction will be considered in Section 4.

As was shown in paper [5], the operator $\mathbf{W}_{\mp}$ is a quasi-Hermitian one. If its eigenvectors form a set $\{\mathbf{d}\}$, then for all nonreal points of its spectrum $\mu\in\sigma\left(\mathbf{W}_{\mp}\right)$ the relation Eq. (11) yields

$$\operatorname{Im}\mu=(\mp)\,\theta_d^2\operatorname{Re}\mu,\ \begin{pmatrix}H\\E\end{pmatrix} \tag{13}$$

where $\theta_d^2=\left\|\mathbf{Pd}^T\right\|^2\left\|\mathbf{Qd}^T\right\|^{-2}$, $\mathbf{Qd}^T\neq 0$.

This result suggests the following. If the working frequency is such that the energy conservation law in the form of Eq. (8), (11) or (12) is valid, then the condition $-1 \notin \sigma(\mathbf{W}_{\mp})$ holds and hence, the following representation occurs

$$\mathbf{S} = (\pm)\frac{\mathbf{I} - \mathbf{W}_{\mp}}{\mathbf{I} + \mathbf{W}_{\mp}}, \begin{pmatrix} H \\ E \end{pmatrix} \tag{14}$$

and the spectral points $\lambda \in \sigma(\mathbf{S})$ satisfy the relation

$$2\,\mathrm{Im}\lambda = \theta_d^2 \left(1 - |\lambda|^2\right), \tag{15}$$

which determines localization of the entire spectrum of the generalized scattering matrix [6].

4. WAVEGUIDE RIGHT ANGLE CORNER BEND

As a classical example of a volumetric discontinuity in the form of a two-port transformer consider a waveguide corner of a fixed height l with a 90°-bend. It is interesting that at least three different approaches to this problem, specifically, the mode-matching technique, the method of partial overlapping regions [10] and the domain-product technique [11], in combination with the technique of matrix operators all lead to the final matrix model in the same form. Here we present solution of this problem by the generalized mode-matching technique with the use of the Green function apparatus.

The whole domain of field determination specified in a single Cartesian frame of coordinates $\{x, y, z\}$ can be divided into three osculating partial sub-regions. These are (1) the semi-infinite waveguide with $\{x \in \Omega_1 \equiv (0, a); z \in (b, \infty)\}$, (2) the same waveguide with $\{x \in (a, \infty); z \in \Omega_2 \equiv (0, b)\}$ and (3) the mode interaction region (or "coupling region" V) with $\{x \in \Omega_1; z \in \Omega_2\}$ (for all these sub-regions $y \in (0, l)$). Two reference planes $\Theta_1 \equiv \{x \in \Omega_1; z = b\}$ and $\Theta_2 \equiv \{x = a; z \in \Omega_2\}$ are coincided with the boundaries of the mentioned partial regions. The normals to these planes outward with respect to the coupling region 3 will be denoted as $\vec{n}_1 = \vec{z}_0$ and $\vec{n}_2 = \vec{x}_0$, respectively.

It is assumed that regions 1 and 2 contain independent sources of waves which generate monochromatic fields (the time-dependent factor is omitted in what follows). Then an electromagnetic wave of a finite power is incident from each waveguide arm upon the discontinuity. The wave field represents an infinite set of modes with any known amplitude distribution ${}^{p}\mathbf{b} \in \ell_2, p = 1,2$.

The diffracting modes are specified by the complete orthonormal systems of transverse eigenfunctions collected in the column-vectors $\boldsymbol{\varphi}_1(x)$, $x \in \Omega_1$, and $\boldsymbol{\varphi}_2(z)$, $z \in \Omega_2$, with the following basic properties

$$\begin{array}{l} \varphi_q^T(\eta)\,\varphi_q(\xi) = \delta(\eta - \xi), \\ \left(\varphi_q, \varphi_q^T\right)_{\Omega_q} = \mathbf{I}, \end{array} \qquad \eta, \xi = \begin{cases} x, x';\ q = 1, \\ z, z';\ q = 2, \end{cases} \tag{16}$$

presented here through the Dirac delta-function and idem-factor **I**. Then, making use of propagation constants of the waveguide modes ${}^{(q)}\gamma_m$, $m = (0),1, \ldots;\ q = 1,2,$ lying in the first quadrant of the complex plane, we form diagonal "matrix operator of similarity" according to the rule

$$\mathbf{I}_{\gamma q} \equiv \left\{ \delta_{mn} \, {}^{(q)}\gamma_m \right\}_{m,n=(0)1}^{\infty},$$

where δ_{mn} is the Kronecker delta. Note especially that the cutoff frequencies $(\exists\, k, m : {}^{(q)}\gamma_m = 0)$ are excluded from the consideration as unphysical values.

Let ${}^{p}U_q = {}^{p}\mathbf{b} \cdot {}^{p}\mathbf{u}_q$ be a phasor to characterize all components of the field in partial region q generated by the source located in port p, $p = 1,2$, $q = \overline{1,3}$. The vector-function ${}^{p}\mathbf{u}_q$ should satisfy the homogeneous Dirichlet (H) and Neumann (E) boundary conditions on the conducting surfaces and condition at infinity for the waveguides, and also should provide finiteness of the field energy within the domain of field determination and continuity of the tangential field components on the boundaries of the osculating partial sub-regions, viz.

$$\begin{cases} {}^{p}\mathbf{u}_1 = {}^{p}\mathbf{u}_3, \\ \partial_z\, {}^{p}\mathbf{u}_1 = \partial_z\, {}^{p}\mathbf{u}_3, \end{cases} \quad x \in \Omega_1,\ z = b; \tag{17}$$

$$\begin{cases} {}^{p}\mathbf{u}_2 = {}^{p}\mathbf{u}_3, \\ \partial_x\, {}^{p}\mathbf{u}_2 = \partial_x\, {}^{p}\mathbf{u}_3, \end{cases} \quad x = a,\ z \in \Omega_2. \tag{18}$$

Mode expansions of the vector-functions of the first two regions on the reference planes can be represented as

$${}^{p}\mathbf{u}_q\Big|_{\Theta_q} = \begin{cases} \left(\mathbf{I} + {}^{p}\mathbf{R}\right)\mathbf{I}_{\gamma p}^{-1/2}\varphi_p, & q = p, \\ {}^{pq}\mathbf{T}\,\mathbf{I}_{\gamma q}^{-1/2}\varphi_q, & q \neq p, \end{cases} \quad p, q = 1,2; \tag{19}$$

$$\left.\frac{\partial\, {}^{p}\mathbf{u}_q}{\partial \vec{n}_q}\right|_{\Theta_q} = \begin{cases} \left(\mathbf{I} - {}^{p}\mathbf{R}\right)\mathbf{I}_{\gamma p}^{1/2}\varphi_p, \, q = p, \\ -{}^{pq}\mathbf{T}\,\mathbf{I}_{\gamma q}^{1/2}\varphi_q, \quad q \neq p. \end{cases} \tag{20}$$

The unknown vector-function for the coupling region 3 is represented using the second Green formula, viz.

$${}^{p}\mathbf{u}_3 = \begin{cases} -\left\{ {}^{p}\mathbf{u}_1, \dfrac{\partial G^D}{\partial \vec{n}_1'} \right\}_{\Omega_1} - \left\{ {}^{p}\mathbf{u}_2, \dfrac{\partial G^D}{\partial \vec{n}_2'} \right\}_{\Omega_2}, (H) \\ \left\{ G^N, \dfrac{\partial\, {}^{p}\mathbf{u}_1}{\partial \vec{n}_1'} \right\}_{\Omega_1} + \left\{ G^N, \dfrac{\partial\, {}^{p}\mathbf{u}_2}{\partial \vec{n}_2'} \right\}_{\Omega_2}, \quad (E) \end{cases} \tag{21}$$

guaranteeing the continuity of the tangential electric field components on the partial region boundaries. In these formulas, $G^{D(N)}(\vec{r},\vec{r}')$ stands for the known Green function of the rectangular coupling region which satisfies the homogeneous Dirichlet (Neumann) boundary conditions. For our purpose there is no need here to write out the explicit form of the function $G^{D(N)}$. Note however that two different sourcewise representations of the same Green function should be used in the two summands of the formula Eq. (21). It remains to secure the continuity of the magnetic field tangential components through substitution of the representation Eq. (21) into the respective equalities Eqs. (17) and (18). As a result we obtain certain functional relations for the vector-functions ${}^{p}\mathbf{u}_1$ and ${}^{p}\mathbf{u}_2$, $p = 1,2$.

Making use of the obtained functional relations and the properties Eq. (16) we find two operator equalities for $p = 1$ and $p = 2$, respectively, through matching the fields on the boundary Ω_1, viz.

$$\begin{aligned} &\mathbf{I} \mp {}^{1}\mathbf{R} = \left(\mathbf{I} \pm {}^{1}\mathbf{R}\right)\mathbf{D}_{11} \pm {}^{12}\mathbf{T}\,\mathbf{D}_{21}; \\ &\mp {}^{21}\mathbf{T} = \pm\, {}^{21}\mathbf{T}\,\mathbf{D}_{11} + \left(\mathbf{I} \pm {}^{2}\mathbf{R}\right)\mathbf{D}_{21}, \end{aligned} \quad \begin{pmatrix} H \\ E \end{pmatrix} \tag{22}$$

and a pair of similar operator relations through matching the fields on the boundary Ω_2, viz.

$$\begin{aligned} &\mp {}^{12}\mathbf{T} = \left(\mathbf{I} \pm {}^{1}\mathbf{R}\right)\mathbf{D}_{12} \pm {}^{12}\mathbf{T}\,\mathbf{D}_{22}; \\ &\mathbf{I} \mp {}^{2}\mathbf{R} = \pm\, {}^{21}\mathbf{T}\,\mathbf{D}_{12} + \left(\mathbf{I} \pm {}^{2}\mathbf{R}\right)\mathbf{D}_{22}. \end{aligned} \quad \begin{pmatrix} H \\ E \end{pmatrix} \tag{23}$$

In the formulas Eqs. (22) and (23) we have introduced the following operator matrices

$$\mathbf{D}=\mathbf{I}_{\Gamma}^{\mp 1/2}\begin{bmatrix}\mathbf{F}_{11} & \mathbf{F}_{12}\\ \mathbf{F}_{21} & \mathbf{F}_{22}\end{bmatrix}\mathbf{I}_{\Gamma}^{\mp 1/2},\begin{pmatrix}H\\ E\end{pmatrix}\qquad \mathbf{I}_{\Gamma}=\begin{bmatrix}\mathbf{I}_{\gamma 1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{\gamma 2}\end{bmatrix}; \tag{24}$$

$$\mathbf{F}_{pq}=\begin{cases}-\left(\left(\dfrac{\partial^2 G^D}{\partial \vec{n}'_p \partial \vec{n}'_q},\varphi_q\right)_{\Omega_q},\varphi_p^T\right)_{\Omega_p}, & (H)\\ \left(\left(G^N,\varphi_p\right)_{\Omega_p},\varphi_q^T\right)_{\Omega_q}. & (E)\end{cases} \tag{25}$$

Also note that the definition Eq. (25) yields the symmetric property $\mathbf{F}_{pq}=\mathbf{F}_{qp}^T$, $p,q=1,2$.

The equalities Eqs. (22) and (23) can be united in an obvious way into the following compact relation

$$\mathbf{I}\mp\mathbf{S}=\left(\mathbf{I}\pm\mathbf{S}\right)\mathbf{D},\ \begin{pmatrix}H\\ E\end{pmatrix}$$

from which the sought-for solution formally follows in the Weyl's form [12]

$$\mathbf{S}=\left(\pm\right)\frac{\mathbf{I}-\mathbf{D}}{\mathbf{I}+\mathbf{D}},\ \begin{pmatrix}H\\ E\end{pmatrix} \tag{26}$$

which is very suitable as for using the symmetric properties $\mathbf{S}^T=\mathbf{S}$ and $\mathbf{D}^T=\mathbf{D}$.

5. CORRECTNESS OF THE OPERATOR MODEL

Consider the problems of existence, uniqueness and robustness of the solution Eq. (26). Comparing the obtained expression with the formula Eq. (14) we find that $\mathbf{D}\equiv\mathbf{W}_{\mp}$. Therefore the problem solution in the form Eq. (26) exists and is unique for all values of the wavenumber except the resonance frequencies of the coupling domain V since under this condition we have $-1\notin\sigma(\mathbf{D})$ as a corollary of the energy conservation law Eq. (11).

As can be shown, the fulfillment of the energy law in the form of Eq. (12) for the problem in question means self-adjointness of the operator matrix $\mathbf{F}$ specified by its elements Eq. (25). In turn, this fact corresponds to the equality $\mathbf{F}_{pq}^*=\mathbf{F}_{pq}$, $p,q=1,2$, which is provided by the property of the traces of the Green function and its second derivative on the reference planes, and also by real-valued transverse eigenfunctions of the hollow rectangular waveguide. And vice-versa, if the Green function of the coupling domain V is not defined (i.e., in the case of resonance frequencies), then the energy conservation law Eq. (12) becomes pointless.

Finally, consider the operator $\mathbf{A} = (\mathbf{I} + \mathbf{D})^{-1}$ which proves to be bounded if the equality Eq. (12) holds. This means that the solution Eq. (26) will be stable for all values of the wavenumber except some vicinity near the resonance frequencies where the condition number $\operatorname{cond}(\mathbf{A}) = \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\|$ becomes enormous.

6. UNIVERSALITY OF THE CONSTRUCTED MATRIX MODEL

Now, let us show that the fundamental energy law for the oscillating power Eq. (6) in the case $V \neq 0$ leads to the operator model in the form of Eq. (26).

We will proceed from the fact that the generalized scattering matrix represents a quasi-Hermitian operator [5] for which, according to the formula Eq. (15), localization of the entire spectrum $\sigma(\mathbf{S})$ is known. Substitution of the eigenvector $\mathbf{d}_\lambda$ of this operator into the relation Eq. (6) yields the equality

$$1 - \lambda^2 = \chi, \tag{27}$$

where $\chi = \frac{1}{\mathbf{d}_\lambda \mathbf{d}_\lambda^T}\left[\left(\nabla_\| U_d, \nabla_\| U_d\right)_V - k^2 \left(U_d, U_d\right)_V \right]$. Solution of the uniformization problem (see, for example, [13]) for the algebraic curve Eq. (27) can be written as

$$\lambda = \frac{\nu - 1}{\nu + 1}, \quad \chi = \frac{4\nu}{(\nu + 1)^2}, \quad \nu \neq -1. \tag{28}$$

As can be shown, this solution alone corresponds to the symmetric property Eq. (5) and this is why it is unique.

So, there exists a single operator of the problem $\mathbf{W} : \{\mathbf{d}\}, \;\; \nu = (1+\lambda)/(1+\lambda)$, $\nu \in \sigma(\mathbf{W})$, related to the sought-for generalized scattering matrix through the Cayley transform as

$$\mathbf{W} = \frac{\mathbf{I} + \mathbf{S}}{\mathbf{I} - \mathbf{S}} \quad \Leftrightarrow \quad \mathbf{S} = \frac{\mathbf{W} - \mathbf{I}}{\mathbf{W} + \mathbf{I}}. \tag{29}$$

The above mentioned symmetric property of the generalized scattering matrix Eq. (5) yields the property $\mathbf{W}^T = \mathbf{W}$ which is tantamount to the representation $\mathbf{W} = \mathbf{W}_0 \mathbf{W}_0^T$, where $\mathbf{W}_0 : h \to h$ is a bounded operator.

Next, let us introduce a matrix operator after the formula $\mathbf{K} = \left(\mathbf{W} + \mathbf{I}\right)^{-1} 2\mathbf{W}_0$. Then the theorem Eq. (6) takes the form

$$\mathbf{b}\,\mathbf{K}\,\mathbf{K}^T\mathbf{b}^T = \left(\nabla_{\|}U, \nabla_{\|}U\right)_V - k^2\left(U, U\right)_V, \forall \mathbf{b} \in h. \tag{30}$$

This suggests that the operator $\mathbf{K}$ determines the oscillating field in the volume V.

Note that the derived expressions

$$\begin{cases} \mathbf{S} = \dfrac{\mathbf{W} - \mathbf{I}}{\mathbf{W} + \mathbf{I}}, \\ \mathbf{K} = \left(\mathbf{W} + \mathbf{I}\right)^{-1} 2\mathbf{W}_0, \end{cases} \qquad \mathbf{S}^2 + \mathbf{K}\,\mathbf{K}^T = \mathbf{I} \tag{31}$$

are similar in their form to the matrix-operator Fresnel formulas obtained in paper [1]. In contrast to the latter they have no scalar analogues. For this reason the equalities Eq. (31) can be referred to as the "generalized operator-based Fresnel formulas".

Thus, the first proposition has been proved, which states the following basic result. *Theorem 1.* For each problem of mode diffraction in a waveguide transformer with the wave interaction region $V \neq 0$ for which the reciprocity theorem Eq. (5) and the oscillating power theorem Eq. (5) are valid there exists a mathematical model in the form of the generalized operator-based Fresnel formulas Eq. (31).

7. CONVERGENCE OF THE PROJECTION APPROXIMATIONS

Now let us find the approximate solution $\hat{\mathbf{S}}$ by the truncation procedure. To that end we will use the orthoprojector $\hat{\mathbf{P}} = \hat{\mathbf{P}}\left(M_1, M_2\right)$ acting from ℓ_2 into a finite-dimensional space whose dimension is determined by the number of the waveguide modes M_1 and M_2 taken into account in the corresponding ports of the wave transformer.

Repeating completely the above reasoning, however now for the approximate representations of the field components in the form of truncated mode expansions, and demanding the energy flow continuity for the field approximation, we obtain the following approximate solution

$$\hat{\mathbf{S}} = \left(\pm\right)\frac{\hat{\mathbf{P}} - \hat{\mathbf{D}}}{\hat{\mathbf{P}} + \hat{\mathbf{D}}} \quad \begin{pmatrix} H \\ E \end{pmatrix} \tag{32}$$

and the energy conservation law

$$\mathrm{Im}\left(\hat{\mathbf{D}}\begin{Bmatrix} \mathbf{U}^{\dagger} \\ \mathbf{U} \end{Bmatrix}\right) = 0. \quad \begin{pmatrix} H \\ E \end{pmatrix} \tag{33}$$

Here we have used the notation $\hat{\mathbf{D}} = \hat{\mathbf{P}}\,\mathbf{D}\,\hat{\mathbf{P}}$.

Consider the operator $\hat{\mathbf{A}} = \left(\hat{\mathbf{P}} + \hat{\mathbf{D}} \right)^{-1}$. If the resonance frequencies of the volume V have been excluded, then the relation Eq. (33) holds and consequently $-1 \notin \sigma\left(\hat{\mathbf{D}}\right)$. Under this condition the family of operators $\left\{ \hat{\mathbf{A}}, \forall M_1, M_2 \right\}$ will be bounded at every point of the domain of their definition. Then, according to the Banach-Steinhaus theorem, this family will be uniformly bounded, $\left\|\hat{\mathbf{A}}\right\| < \text{const}$, $\forall M_1, M_2$. Therefore,

$$\text{cond}\left(\hat{\mathbf{A}}\right) < \text{const} \cdot \left(1 + \|\mathbf{D}\|\right) < \infty. \tag{34}$$

Thus, the projection approximation Eq. (32) is existent, unique and stable if the operation frequency lies outside the nearest vicinity of the resonance frequency of the coupling region V where $\text{cond}\left(\hat{\mathbf{A}}\right) \to \infty$.

Next, the formulas Eqs. (26) and (32) yield the representation

$$\hat{\mathbf{P}}\mathbf{S} - \hat{\mathbf{S}} = 2\,\hat{\mathbf{A}}\left(\hat{\mathbf{P}}\,\mathbf{D} - \hat{\mathbf{D}} \right)\mathbf{A}, \tag{35}$$

using which we find the estimate

$$\left\| \left(\hat{\mathbf{P}}\mathbf{S} - \hat{\mathbf{S}} \right) \mathbf{b}^T \right\| < \text{const}_1 \left\| \left(\mathbf{I} - \hat{\mathbf{P}} \right) \mathbf{d}^T \right\|, \tag{36}$$

where $\mathbf{d} = 2\mathbf{b}\,\mathbf{A}$ and $\forall \mathbf{b} \in \ell_2$. Since the orthoprojector $\hat{\mathbf{P}}\left(M_1, M_2\right)$ in the space ℓ_2 strongly (but non-uniformly) converges to the identity operator with $M_1, M_2 \to \infty$, $\forall M_1 / M_2$, the obtained estimate Eq. (36) means that the following proposition is valid.
Theorem 2. The projection approximations Eq. (32) strongly converge to the true solution Eq. (26), with the relative convergence phenomenon being absent.

8. CONCLUSIONS

The generalized mode-matching technique has been applied to analyzing wave transformers with a resonant mode-interaction cavity $V \neq 0$. This class of the diffraction problems is defined as such for which the four basic energy laws in the form of Eqs. (5) to (7) are valid.

The solution of the classical problem on the waveguide right angled bend has been obtained in the form of the generalized Fresnel formula for the scattering matrix $\mathbf{S}$ in the operator form Eq. (26).

It has been proven that such form of the matrix-operator model of the mode-matching technique is common for the whole class of the waveguide mode diffraction problems under consideration. The universality of the mentioned Fresnel formula is conditioned by the energy law Eq. (6) and existence of a single operator of the problem which is determined by the geometry of volumetric discontinuity and dependence on working frequency.

It has been proven rigorously that correctness of the generalized operator-based Fresnel formulas is an immediate corollary of the energy conservation law in the form Eqs. (11) and (12).

The applicability of truncation procedure for determining approximation of the generalized scattering matrix has been justified. The unconditional strong convergence of the projection approximations $\hat{\mathbf{S}}$ has been proven analytically.

The developed and rigorously substantiated generalized mode-matching technique can be efficiently used for strict analysis of microwave transformers.

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