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MATHEMATICAL METHODS IN ELECTROMAGNETIC THEORY

GENERALIZED MODE-MATCHING TECHNIQUE IN THE THEORY OF GUIDED WAVE DIFFRACTION. PART 1: FRESNEL FORMULAS FOR SCATTERING OPERATORS*

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A generalization of the conventional mode-matching technique corresponding to a new formulation of the problem of wave diffraction by waveguide discontinuities is presented. The matrix-operator formalism used in the study for the modal analysis is briefly described. Fresnel formulas are derived for the sought-for operators of mode reflection and transmission in the canonical problem of step discontinuity in the guide. The correctness of the found matrix-operator model is proved analytically. It is shown that the obtained results are valid for a wide class of scalar problems of wave diffraction by step-like discontinuities in the waveguide. The developed generalization of the mode-matching technique is intended for efficient and rigorous analysis of waveguide units and microwave devices.

KEY WORDS: *mode-matching technique, Cayley transformation, Fresnel formulas, scattering operator*

1. INTRODUCTION

A paradoxical situation has arisen in the computational electrodynamics long ago concerning the use of the mode-matching technique (also known as the method of the partial contiguous regions) for solving problems of wave diffraction by a discontinuity in the waveguide. On the one hand, this technique proves to be in demand and

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apparently the most widespread since it is rather universal, relatively simple to implement and allows obtaining numerical results which are quite acceptable for the modern engineering practice. While on the other hand, no satisfactory substantiation of this numerical-analytical method in the general case has been suggested by the present time which means that application of the mode-matching technique to the majority of practically important problems differs but slightly from a heuristic approach.

As is known, the classical mode-matching technique always leads to the mathematical model in the form of an infinite system of linear algebraic equations. The recent, most appreciable surge of interest toward studying the mode-matching technique falls on the period when the phenomenon of the relative convergence of approximate solutions to such matrix equation had been demonstrated both analytically and numerically, first for the special problem on waveguide bifurcation (see book [1] and references therein) and then for other problems as well [2,3]. However, the further efforts undertaken to prove the existence, uniqueness and robustness of the solution to these systems of linear algebraic equations of the mode-matching technique and determining convergence conditions for projection approximations have not yielded any significant new results [4].

In our opinion the evident stagnation in the development of the theory of the mode-matching technique during a long time indicates a basic unsuitability of the matrix model in the form of an infinite system of linear algebraic equations for solving the problem of validation of this method.

We assert that the infinite systems of linear algebraic equations with respect to the vector of unknown coefficients of the (generalized) Fourier series representing the field phasor are not intrinsic of the mode-matching technique *per se*. They appear only in connection with a certain and as it seems rather special formulation of the mode diffraction problem.

The conventional statement of the problem of mode diffraction by a waveguide discontinuity is as follows. A specified single waveguide mode is scattered by a discontinuity and it is necessary to find amplitudes of the excited modes (both propagating and evanescent ones).

A corollary of the problem formulation like this is all the above mentioned mathematical difficulties which can be eliminated through changing the problem statement to the following (more natural in our opinion). Let an electromagnetic wave of a finite power be incident upon a discontinuity. The incident field consists of an infinite set of modes with any known amplitude distribution. It is necessary to find the scattering operators.

If the diffraction problem is posed as suggested above, then application of the mode-matching technique yields an equation with respect to the scattering operator rather than an infinite system of linear algebraic equations [5-8]. The method of introducing this scattering operator consists in replacing the unknown vector of the Fourier coefficients belonging to the space of sequences H, $\mathbf{x} \in H$, by the sough-for matrix operator $\mathbf{X}: H \to H$ and will be referred to in what follows as the matrix operator technique. In the applied electrodynamics this approach had appeared to be consistently implemented for the first time in the work [9].

For the canonical problem on a uniform bend of a rectangular waveguide it has been found [5] that this new approach naturally leads to the Fresnel formulas

$$\mathbf{R} = \frac{\mathbf{D}\mathbf{D}^T - \mathbf{I}}{\mathbf{D}\mathbf{D}^T + \mathbf{I}}; \quad \mathbf{T} = \left(\mathbf{D}\mathbf{D}^T + \mathbf{I}\right)^{-1} 2\mathbf{D} \tag{1}$$

for matrix operators of reflection $\mathbf{R}$ and transmission $\mathbf{T}$ (here the given operator $\mathbf{D}$ is determined by the geometry of the problem and depended on the field frequency ω). Later this result has been disseminated on the problem of mode diffraction by other kinds of waveguide discontinuities [6-8].

This paper (in three parts) is aimed at briefly presenting new results of the theory of the mode-matching technique generalized through introducing matrix operators as applied to those problems which leads to the operator-based Fresnel formulas including a rigorous substantiation of the correctness of the matrix model Eq. (1) (the present paper), an analytical proof of the applicability of the truncation procedure for determining approximate solutions and a factorization of the generalized scattering matrix (the papers are under preparation for publication).

Note that the presented approach can be also regarded as a further development of the spectral scattering operator method [10-12].

2. SCALAR MODAL ANALYSIS

Let the field in a simple (partial) region V be completely determined by the scalar phasor $U = U(\omega, \vec{r})$ which is dependent on the radius-vector $\vec{r} \in V$ and satisfies the wave equation

$$\Delta U + k^2 U = \phi, \tag{2}$$

where $k = \sqrt{\varepsilon\mu}\,\omega$ is the wavenumber, $\operatorname{Im} k = 0$, and the function $\phi = \phi(\omega, \vec{r})$ specifies the field source. A region will be referred to as simple if it allows solution of a given boundary-value problem for the Helmholtz equation Eq. (2) by the variable separation method in a suitable coordinate frame. Here we assume that the region V is represented by a finite or semi-infinite section of a regular waveguide.

Let us write the ordinary expansion of the phasor in waveguide modes in the form

$$U = \sum_{m=1}^{\infty} x_m \varphi_m(\vec{r}) = \mathbf{x} \cdot \boldsymbol{\varphi}(\vec{r}), \tag{3}$$

where $\mathbf{x} = \{x_m\}_{m=1}^{\infty}$ is the row-vector of the sought-for complex coefficients to be determined and $\boldsymbol{\varphi}(\vec{r}) = \{\varphi_m(\vec{r})\}_{m=1}^{\infty}$ is the column-vector of the known functions representing modes of the given regular waveguide. In what follows we will consider the “normal modes” of the form

$$\varphi_m(\vec{r}) = \psi_m(\vec{r}_\perp)\exp(\gamma_m\zeta), \quad m = \overline{1,\infty}. \tag{4}$$

Here $\psi_m(\vec{r}_\perp)$ denotes the eigenfunction of a homogeneous boundary-value problem for a transverse cross-section of the regular waveguide, $\vec{r}_\perp \in \Theta$, and γ_m is the propagation constant of m-th mode along the longitudinal axis $O\zeta$ of the waveguide. The regular representation Eq. (4) makes it possible to determine the structure of $\boldsymbol{\varphi}(\vec{r})$ in a matrix-vector form, viz.

$$\boldsymbol{\varphi}(\vec{r}) = \mathbf{E}(\zeta)\boldsymbol{\psi}(\vec{r}_\perp). \tag{5}$$

Here we have introduced a diagonal matrix operator $\mathbf{E}(\zeta) \equiv \{\delta_{mn}\exp(\gamma_m\zeta)\}$ such that $\mathbf{E}(0) = \mathbf{I}$ is an identity operator, and also a column-vector of real-valued transverse eigenfunctions $\boldsymbol{\psi}(\vec{r}_\perp)$, whose basic properties are defined by the following equalities

$$\boldsymbol{\psi}^T(\vec{r}_\perp)\boldsymbol{\psi}(\vec{r}\,'_\perp) = \delta(\vec{r}_\perp - \vec{r}\,'_\perp); \quad (\boldsymbol{\psi}, \boldsymbol{\psi}^T)_\Theta = \mathbf{I}. \tag{6}$$

Here we have used the conventional notation for the Dirac delta-function and scalar (bilinear) product of functions, and superscript “T” stands for the transposition. Making use of the formulas Eqs. (3) and (5) the derivative of the phasor along the waveguide axis $O\zeta$ can be represented in the form

$$\partial_\zeta U = \mathbf{x}\cdot\partial_\zeta\boldsymbol{\varphi}(\vec{r}); \quad \partial_\zeta\boldsymbol{\varphi}(\vec{r}) = \mathbf{E}_\gamma(\zeta)\boldsymbol{\psi}(\vec{r}_\perp), \tag{7}$$

where $\mathbf{E}_\gamma(\zeta) \equiv \{\delta_{mn}\gamma_m\exp(\gamma_m\zeta)\}$. To simplify writing further mathematical manipulation we will use the notation $\mathbf{I}_\gamma \equiv \mathbf{E}_\gamma(0)$.

The flux of the oscillating power through the transverse cross-section of the waveguide within the plane $\zeta = 0$ can be written up to inessential factors as

$$F_{osc} = [(U, \partial_\zeta U)_\Theta]_{\zeta=0} = \mathbf{x}\,\mathbf{I}_\gamma\mathbf{x}^T = \mathbf{a}\cdot\mathbf{a}^T, \tag{8}$$

where $\mathbf{a} = \mathbf{x}\,\mathbf{I}_\gamma^{1/2}$. The complex power flux through the same cross-section Θ is determined by the value

$$F_{osc} = \left[(U, \partial_\zeta U^*)_\Theta\right]_{\zeta=0} = \mathbf{x}\,\mathbf{I}_\gamma^*\mathbf{x}^\dagger = \mathbf{a}\,\mathbf{U}\,\mathbf{a}^\dagger, \tag{9}$$

where the asterisk " $*$ " and the dagger " $\dagger$ " mean the complex and Hermitian conjugations, respectively, and a $\mathbf{U} \equiv \mathbf{I}_{\gamma}^{-1/2}\, \mathbf{I}_{\gamma}^{*}\, \mathbf{I}_{\gamma^*}^{-1/2} = \left\{ \delta_{mn} \exp\left[-i \arg\left(\gamma_m\right)\right]\right\}$ is the operator of the waveguide port (or portal operator) which is uniquely defined under the condition $\gamma_m \neq 0, \forall m.$

Let P modes be propagating for a working frequency in the given waveguide. Consider the orthoprojectors

$$\mathbf{P} = \left\{ \sum_{p=1}^{P} \delta_{mp}\, \delta_{pn} \right\} \text{ and } \mathbf{Q} = \left\{ \sum_{q=P+1}^{\infty} \delta_{mq}\, \delta_{qn} \right\}; \tag{10}$$

such that $\mathbf{a}_- = \mathbf{a}\,\mathbf{P}$ and $\mathbf{a}_+ = \mathbf{a}\,\mathbf{Q}$ are vectors of the amplitudes of the propagating and evanescent modes, respectively. Also note that the portal operator is a unitary one, $\mathbf{U}^{-1} = \mathbf{U}^{\dagger}$, and it follows from the representation $\mathbf{U} = \mathbf{Q} \mp i\,\mathbf{P}$ that its numerical range lies completely in the forth or first quadrant of the complex plane in accordance with the selected time flow direction $\exp(\pm i\omega t)$.

Finally, let us postulate that the complex power flux through the waveguide,

$$F_{cmp} = \left\|\mathbf{a}_+\right\|_{\ell_2}^2 \mp i\, \left\|\mathbf{a}_-\right\|_{\ell_2}^2 ,$$

is a bounded quantity, $\left|F_{cmp}\right| < \infty$ (at that we automatically obtain the condition $\left|F_{osc}\right| < \infty$). This condition is equivalent to the requirement $\mathbf{a} \in \ell_2 \Leftrightarrow \mathbf{x} \in \widetilde{\ell}_2$, where $\ell_2 \equiv \left\{ \mathbf{a} : \sum_{m=1}^{\infty} \left|a_m\right|^2 < \infty \right\}$ and $\tilde{\ell}_2 \equiv \left\{ \mathbf{x} : \sum_{m=1}^{\infty} m \left|x_m\right|^2 < \infty \right\}$ are Hilbert spaces of sequences of complex numbers.

3. MATRIX-OPERATOR FORMALISM

Now, we will set forth the fundamentals of the matrix operator technique as applied to scalar problems of the theory of stationary diffraction of the waveguide modes.

According to the new formulation of the diffraction problem let us represent the solution of the Helmholtz equation Eq. (2) in the form of a series, viz.

$$U = \sum_{m=1}^{\infty} b_m\, u_m\left(\vec{r}\,\right) = \mathbf{b} \cdot \mathbf{u}\left(\vec{r}\,\right), \tag{11}$$

where the row-vector $\mathbf{b} = \left\{ b_m \right\}_{m=1}^{\infty}$ is given and $\mathbf{u}\left(\vec{r}\right) = \left\{ u_m\left(\vec{r}\right)\right\}_{m=1}^{\infty}$ is the column-vector of the functions to be determined (compare with the representation Eq. (3)). In view of the known arbitrariness of specifying the vector **b** the standard formulation of the electrodynamic problem is transferred onto the function $\mathbf{u}(\vec{r})$ which should satisfy, in particular, the Helmholtz equation and the given boundary conditions. Therefore, in each regular waveguide for each unknown function $u_m(\vec{r})$, $m = 1,2,...$, we have the usual expansion in waveguide modes in accordance with the superposition principle for the fields of incident, reflected and transmitted waves.

In other words the representation of the considered phasor U in the form of the decomposition Eq. (11) is equivalent to the replacing the Fourier coefficients $\{x_m\}_{m=1}^{\infty}$ in the mode expansion of the field Eq. (3) in elements of an infinite matrix whose physical meaning is the scattering operator. So, the sought-after unknowns are the operators of mode reflection and transmission.

The vector space for **b** and the function space for $u_m(\vec{r})$, $m = 1,2,...$, should be selected such that the initial phasor Eq. (11) will belong to the Sobolev space $H^1(V)$. It proves that the all obtained formulas are characterized by the maximum simplicity and symmetry when $\mathbf{b} \in \ell_2$. In this case it is necessary to standardize the sought-for scattering operators to the form $\mathbf{R}: \ell_2 \to \ell_2$ and $\mathbf{T}: \ell_2 \to \ell_2$ (see, for example, [5]).

According to the matrix operator technique let us introduce the representation $\mathbf{a} = \mathbf{b}\mathbf{L}$, where $\mathbf{L}: \ell_2 \to \ell_2$ is a suitable scattering operator (specific implementations of this operator will be presented in the next Section). Then, according to the relation Eq. (8) the oscillating power flux through the plane $\zeta = 0$ is proportional to the value

$$F_{osc} = \left[\left(U, \partial_\zeta U\right)_\Theta\right]_{\zeta=0} = \mathbf{b}\mathbf{L}\mathbf{L}^T\mathbf{b}^T. \tag{12}$$

Note that the figuring here operator $\mathbf{L}\mathbf{L}^T$ shows a characteristic symmetry with respect to the transposition. The formula Eq. (9) for the complex power flux through the same cross-section yields

$$F_{cmp} = \left[\left(U, \partial_\zeta U^*\right)_\Theta\right]_{\zeta=0} = \mathbf{b}\mathbf{L}\mathbf{U}\mathbf{L}^\dagger\mathbf{b}^\dagger. \tag{13}$$

Taking into account the properties of the portal operator, we find that the numerical range of the operator $\mathbf{L}\mathbf{U}\mathbf{L}^\dagger$ lies within the forth or first quadrant of the complex plane. These properties of the operators in the relations Eqs. (12) and (13) play the key role in the further analysis.

4. THE OPERATOR-BASED FRESNEL FORMULAS

Consider the canonical problem of scattering the LM_{m0}, $m = 1,2,...$ or LE_{m1}, $m = 0,1,...$ modes by a step discontinuity in a hollow infinite rectangular waveguide with perfectly conducting walls specified in a Cartesian coordinate frame. Geometrically the domain of field definition $\{x\in\Omega_s,\ s=1,2;\ y\in(0,l);\ z\in(-\infty,\infty)\}$ can be separated into two contiguous partial regions with transverse cross-sections $\Omega_1\times l$ and $\Omega_2\times l$. Without loss of generality we will assume in what follows that $\Omega_1=\Omega_2\cup\Omega'$. The reference plane is coincided with the discontinuity aperture $\{x\in\Omega_2;\ y\in(0,l);z=0\}$. The time dependence is assumed to be $\exp(i\omega t)$.

Let the scalar function ${}^pU_q(x,z)$ stands for the phasor determining all components of the electromagnetic field in waveguide q, whose source is located in waveguide p ($p,q=1,2$). The field of this source contains the complete spectrum of modes $\{LM_{m0}\}_{m=1}^{\infty}$ or $\{LE_{m1}\}_{m=0}^{\infty}$ with any known distribution of amplitudes collected in the row-vector ${}^p\mathbf{b}=\{{}^pb_m\}_{m=(0)1}^{\infty}\in\ell_2$.

The continuity condition for the tangential electric and magnetic field components on the reference plane written in the form

$$\left\{\begin{aligned}&\left.\begin{aligned}&{}^pU_1={}^pU_2,\\&\partial_z{}^pU_1=\partial_z{}^pU_2,\end{aligned}\right\}\quad x\in\Omega_2,\\&\left.\begin{aligned}&{}^pU_1=0,\quad (LM)\\&\partial_z{}^pU_1=0,\ (LE)\end{aligned}\right\}\quad x\in\Omega',\end{aligned}\right.\qquad z=0,\ p=1,2,$$

bring us to the implication

$$\begin{cases}{}^p\mathbf{b}\left({}^p\mathbf{u}_1-{}^p\mathbf{u}_2\right)=0,\\{}^p\mathbf{b}\,\partial_z\left({}^p\mathbf{u}_1-{}^p\mathbf{u}_2\right)=0,\end{cases}\quad\forall\,{}^p\mathbf{b}\in\ell_2\Rightarrow\begin{cases}{}^p\mathbf{u}_1={}^p\mathbf{u}_2,\\\partial_z{}^p\mathbf{u}_1=\partial_z{}^p\mathbf{u}_2,\end{cases}\qquad x\in\Omega_2,\ z=0.\tag{14}$$

The key point here is that the vector ${}^p\mathbf{b}$ is common for the two partial regions. Similar reasoning leads us to the homogeneous boundary conditions

$$\begin{aligned}&{}^p\mathbf{u}_1=0,\quad (LM)\\&\partial_z{}^p\mathbf{u}_1=0,\ (LE)\end{aligned}\quad x\in\Omega',\ z=0\tag{15}$$

on the step face.

The mode expansions for the functions under consideration on the reference plane are as follows $(p,q=1,2)$

$${}^p\mathbf{u}_q(x,0)=\begin{cases}\left(\mathbf{I}+{}^p\mathbf{R}\right)\mathbf{I}_{\gamma p}^{-1/2}\boldsymbol{\psi}_p(x),q=p,\\{}^{pq}\mathbf{T}\,\mathbf{I}_{\gamma q}^{-1/2}\boldsymbol{\psi}_q(x),\qquad q\neq p,\end{cases}x\in\Omega_2;\tag{16}$$

$$\left(\partial_z {}^p\mathbf{u}_q\right)(x,0)=\begin{cases}(\mp)\left(\mathbf{I}-{}^p\mathbf{R}\right)\mathbf{I}_{\gamma p}^{1/2}\boldsymbol{\psi}_p(x), q=p,\\ (\mp)\,{}^{pq}\mathbf{T}\,\mathbf{I}_{\gamma q}^{1/2}\boldsymbol{\psi}_q(x), \quad q\neq p,\end{cases} \quad p=\begin{Bmatrix}1\\2\end{Bmatrix}, \quad x\in\Omega_2. \quad (17)$$

Substitution of the expressions Eqs. (16) and (17) into Eq. (14) yields the following set of matrix equalities for $x\in\Omega_2$

$$\begin{cases}\left(\mathbf{I}+{}^p\mathbf{R}\right)\mathbf{I}_{\gamma p}^{-1/2}\boldsymbol{\psi}_p(x)={}^{pq}\mathbf{T}\,\mathbf{I}_{\gamma q}^{-1/2}\boldsymbol{\psi}_q(x),\\ \left(\mathbf{I}-{}^p\mathbf{R}\right)\mathbf{I}_{\gamma p}^{1/2}\,\boldsymbol{\psi}_p(x)={}^{pq}\mathbf{T}\,\mathbf{I}_{\gamma q}^{1/2}\boldsymbol{\psi}_q(x),\end{cases} \quad (18)$$

while from the boundary conditions Eq. (15) we found

$$\begin{cases}\left(\mathbf{I}\pm{}^1\mathbf{R}\right)\mathbf{I}_{\gamma_1}^{\mp 1/2}\boldsymbol{\psi}_1(x)=0,\\ {}^{21}\mathbf{T}\,\mathbf{I}_{\gamma_1}^{\mp 1/2}\boldsymbol{\psi}_1(x)=0,\end{cases} x\in\Omega'. \quad \begin{pmatrix}LM\\LE\end{pmatrix} \quad (19)$$

By applying the Galerkin procedure to the relations Eqs. (18) and (19) we formally obtain the desired solution

$$\begin{cases}{}^1\mathbf{R}=(\pm)\left(\mathbf{D}_1-\mathbf{I}\right)\left(\mathbf{D}_1+\mathbf{I}\right)^{-1}, \\ {}^{12}\mathbf{T}=\left(\mathbf{D}_1+\mathbf{I}\right)^{-1}2\,\mathbf{D}_0,\end{cases} \begin{pmatrix}LM\\LE\end{pmatrix}$$
$$\begin{cases}{}^2\mathbf{R}=(\mp)\left(\mathbf{D}_2-\mathbf{I}\right)\left(\mathbf{D}_2+\mathbf{I}\right)^{-1}, \\ {}^{21}\mathbf{T}=\left(\mathbf{D}_2+\mathbf{I}\right)^{-1}2\,\mathbf{D}_0^T.\end{cases} \begin{pmatrix}LM\\LE\end{pmatrix} \quad (20)$$

Here we have used the notation

$$\mathbf{D}_0=\mathbf{I}_{\gamma_1}^{\pm 1/2}\left(\boldsymbol{\psi}_1,\boldsymbol{\psi}_2^T\right)_{\Omega_2}\mathbf{I}_{\gamma_2}^{\mp 1/2}, \quad \begin{pmatrix}LM\\LE\end{pmatrix}$$
$$\mathbf{D}_1=\mathbf{D}_0\mathbf{D}_0^T, \quad \mathbf{D}_2=\mathbf{D}_0^T\mathbf{D}_0. \quad (21)$$

The derived solution Eq. (20) represents the Fresnel formulas for the operators of waveguide mode reflection and transmission.

The existence of the inverse operators in the solution Eq. (20) follows from the conservation law for the complex power and will be proven rigorously in Section 6. Here we will mention the symmetry properties of the scattering operators which emerge from the form of the obtained solution. Indeed, from the first Fresnel formula we obtain

$$\mathbf{R}_p = \mathbf{I} - 2\left(\mathbf{D}_p + \mathbf{I}\right)^{-1} = \mathbf{R}_p^T,$$
$$\mathbf{R}_p = \begin{cases} (\pm)\,{}^1\mathbf{R}, & p = 1, \\ (\mp)\,{}^2\mathbf{R}, & p = 2, \end{cases} \quad \begin{pmatrix} LM \\ LE \end{pmatrix} \tag{22}$$

since $\mathbf{D}_p^T = \mathbf{D}_p$, $p = 1, 2$ by the definition Eq. (21). The symmetry property of the transmission operator ${}^{pq}\mathbf{T}^T = {}^{qp}\mathbf{T}$ is checked by the direct substitution.

Note that the first operator-based Fresnel formula in Eq. (20) is also known as the Cayley transformation [13] (in what follows we will use both these names as equivalent). Provided the existence condition for this transform is met, $-1 \notin \sigma\left(\mathbf{D}_p\right)$, it proves to be invertible, viz.

$$\mathbf{R}_p = \frac{\mathbf{D}_p - \mathbf{I}}{\mathbf{D}_p + \mathbf{I}} \quad \Leftrightarrow \quad \mathbf{D}_p = \frac{\mathbf{I} + \mathbf{R}_p}{\mathbf{I} - \mathbf{R}_p}. \tag{23}$$

Here the Cayley transforms have been written in the Weyl's form [13].

5. RECIPROCITY PRINCIPLE AND POWER CONSERVATION LAW IN THE OPERATOR FORM

In the phasor domain, four basic energy laws should be taken into account [14-16]. These are the first and second Lorentz theorems [15], the theorem of the oscillating power [17] and the well-known complex power theorem. For the problem under consideration these theorems yield the presented below four groups of operator equalities which determine the basic properties of the scattering operators [5,8,14]. In the course of derivation of these equalities the fundamental property of the space ℓ_2 is used; namely, each operator is unambiguously determined by own quadratic form.

Thus, the continuity condition for the oscillating power flux through the discontinuity aperture and the first Lorentz lemma yield four matrix relations as follows

$$\left({}^p\mathbf{u}_1, \partial_z\, {}^p\mathbf{u}_1^T\right)_{\Omega_1} = \left({}^p\mathbf{u}_2, \partial_z\, {}^p\mathbf{u}_2^T\right)_{\Omega_2}, \; p, q = 1, 2,$$
$$\left({}^p\mathbf{u}_1, \partial_z\, {}^q\mathbf{u}_1^T\right)_{\Omega_1} = \left({}^p\mathbf{u}_2, \partial_z\, {}^q\mathbf{u}_2^T\right)_{\Omega_2}, \; q \neq p. \tag{24}$$

Combining these equalities with the representations Eqs. (16), (17) and (19) and making use of the orthogonal property of the transverse eigenfunctions Eq. (6) we arrive at

$${}^p\mathbf{R}^T = {}^p\mathbf{R}, \quad {}^{pq}\mathbf{T}^T = {}^{qp}\mathbf{T}; \tag{25}$$

$$ {}^{p}\mathbf{R}^{2} + {}^{pq}\mathbf{T}\,{}^{pq}\mathbf{T}^{T} = \mathbf{I}; \tag{26} $$

$$ {}^{p}\mathbf{R}\,{}^{pq}\mathbf{T} + \left({}^{q}\mathbf{R}\,{}^{qp}\mathbf{T}\right)^{T} = 0. \tag{27} $$

Usually these operator relations are written in the form of the symmetry $\mathbf{S}^{T} = \mathbf{S}$ and involutivity $\mathbf{S}^{2} = \mathbf{I}$ (or $\mathbf{S}^{-1} = \mathbf{S}$) of the generalized scattering matrix. The latter property means that the operators $\left(\mathbf{I} \pm \mathbf{S}\right)/2$ are projectors (but not orthoprojectors), whence it follows that the spectrum $\sigma\left(\mathbf{S}\right)$ consists of only two points $\{+1, -1\}$ of infinite multiplicity and $\mathbf{S}$ is the symmetry (but not canonical symmetry) of the space ℓ_2. As a result, the generalized immitance (i.e., the Cayley transform of the operator **S**) for step-like discontinuities is indefinable [14].

Next, the corollaries of the continuity of the complex power flux through the discontinuity aperture and the second Lorentz lemma will be the following equalities

$$ \begin{aligned} &\left({}^{p}\mathbf{u}_1, \partial_z\, {}^{p}\mathbf{u}_1^{\dagger}\right)_{\Omega_1} = \left({}^{p}\mathbf{u}_2, \partial_z\, {}^{p}\mathbf{u}_2^{\dagger}\right)_{\Omega_2}, \; p, q = 1, 2, \\ &\left({}^{p}\mathbf{u}_1, \partial_z\, {}^{q}\mathbf{u}_1^{\dagger}\right)_{\Omega_1} = \left({}^{p}\mathbf{u}_2, \partial_z\, {}^{q}\mathbf{u}_2^{\dagger}\right)_{\Omega_2}, \; q \neq p. \end{aligned} \tag{28} $$

Whence it follows that

$$ \left(\mathbf{I} + {}^{p}\mathbf{R}\right)\mathbf{U}_p\left(\mathbf{I} - {}^{p}\mathbf{R}^{\dagger}\right) = {}^{pq}\mathbf{T}\,\mathbf{U}_q\,{}^{pq}\mathbf{T}^{\dagger}; \tag{29} $$

$$ {}^{pq}\mathbf{T}\,\mathbf{U}_q\left(\mathbf{I} - {}^{q}\mathbf{R}^{\dagger}\right) = \left(\mathbf{I} + {}^{p}\mathbf{R}\right)\mathbf{U}_p\,{}^{qp}\mathbf{T}^{\dagger}. \tag{30} $$

The operator relations Eqs. (29) and (30) are combined in a simple equality for the characteristic scattering operator [16]

$$ \mathbf{G} \equiv \left(\mathbf{I} + \mathbf{S}\right)\mathbf{U}\left(\mathbf{I} - \mathbf{S}^{\dagger}\right) = 0, \; \mathbf{U} = \operatorname{diag}\left(\mathbf{U}_1, \mathbf{U}_2\right). \tag{31} $$

The obtained operator form Eq. (31) of the energy conservation law is valid for all problems of wave diffraction by step-like waveguide discontinuities [14,16].

As will shown in what follows, the relations Eqs. (25) through (27) make it possible to extend the operator Fresnel formulas to cover the entire class of mode diffraction problems for the planar two-port waveguide transformers. Whereas the power conservation law in the operator form Eqs. (29) to (31) guarantees the existence, uniqueness and robustness of the obtained solution Eq. (20).

6. UNIVERSALITY OF THE OPERATOR MODEL

The operator-based Fresnel formulas Eq. (20) identically satisfy the equations Eqs. (25) to (27) which can be easily verified through direct substitution. Now, we will show that in their turn these power relations, which are valid for the entire class of problems under consideration, lead to the operator-based Fresnel formulas.

To this end let us write the equality Eq. (26) in the form

$$\left(\mathbf{I}+\mathbf{R}\right)\left(\mathbf{I}-\mathbf{R}\right)^{T}=\mathbf{T}\,\mathbf{T}^{T} \tag{32}$$

and will treat it as an equation with respect to the operators of mode reflection and transmission. In this Section to simplify writings the superscripts p,q are omitted. It follows from this equation that the spectral points $\lambda\in\sigma\left(\mathbf{R}\right)$ and $\tau\in\sigma\left(\mathbf{T}\mathbf{T}^{T}\right)$ belong to the algebraic curve

$$\lambda^{2}+\tau=1,\quad \lambda,\tau\in\mathbb{C}, \tag{33}$$

which allows a known solution to the uniformization problem in the form of rational functions (see, for example, [18]). This solution can be written as

$$\lambda=\frac{t-1}{t+1},\quad \tau=\frac{4t}{\left(t+1\right)^{2}},\quad t\neq-1\,. \tag{34}$$

Based on the spectrum mapping theorem (see, for example, [19]) we conclude that there exists an operator $\mathbf{D}$ such that $t=\left(1+\lambda\right)/\left(1-\lambda\right)\in\sigma\left(\mathbf{D}\right)$ and the following representation is valid

$$\mathbf{R}=\frac{\mathbf{D}-\mathbf{I}}{\mathbf{D}+\mathbf{I}}\quad\Rightarrow\quad\begin{cases}\mathbf{I}+\mathbf{R}=2\left(\mathbf{D}+\mathbf{I}\right)^{-1}\mathbf{D};\\ \mathbf{I}-\mathbf{R}=2\left(\mathbf{D}+\mathbf{I}\right)^{-1}.\end{cases} \tag{35}$$

Taking into account the symmetry property of the reflection operator Eq. (25) we obtain $\mathbf{D}=\widetilde{\mathbf{D}}_{0}\widetilde{\mathbf{D}}_{0}^{T}$, where $\widetilde{\mathbf{D}}_{0}:\ell_{2}\to\ell_{2}$ is so far an arbitrary bounded matrix operator. Then the equality Eq. (32) with account of the relations Eq. (35) yields the second Fresnel formula

$$\mathbf{T}=\left(\mathbf{D}+\mathbf{I}\right)^{-1}2\,\mathbf{D}_{0}. \tag{36}$$

Here we have used the notation $\mathbf{D_0}=\widetilde{\mathbf{D}}_{\mathbf{0}}\mathbf{C}$, where the second multiplier possesses the property $\mathbf{C}\mathbf{C}^{T}=\mathbf{I}$, such that it can be immediately set $\mathbf{D}=\mathbf{D}_{0}\mathbf{D}_{0}^{T}$. Now, the arbitrary bounded matrix operator $\mathbf{D}_{0}$ should be redefined by the complex power conservation law Eq. (31), what will be done in Section 7. Note that it seems impossible to distribute the rational functions in the equalities Eq. (34) in a different way since it then leads to violation of the relations Eqs. (25) and (27).

Thus, the possibility in principle of parameterization of the curve Eq. (33) with the use of the one-valued functions Eq. (34) in this case guarantees the existence of a single operator of the problem, which completely determines the laws of mode reflection and transmission. So, we have proved the

Theorem 1. Each problem of mode diffraction by step-like waveguide discontinuity for which the reciprocity theorem and the oscillating power theorem hold in the form of the equalities Eqs. (25) through (27) can be mathematically modeled by the operator-based Fresnel formulas Eqs. (35) and (36).

7. CORRECTNESS OF THE OPERATOR-BASED FRESNEL FORMULAS

In terms of the Cayley transform $\mathbf{D}_p$ the power conservation law in the forms Eqs. (29) to (31) takes an especially simple form which allows investigating the basic properties of this operator and, in doing so, proving the correctness of the matrix-operator model in the form of the Fresnel formulas Eq. (20).

First, it results from the formula Eq. (31) that all operator representations of the complex power conservation law follow from the common condition

$$\mathbf{D}_0 = \begin{Bmatrix} \mathbf{U}_1^\dagger \\ \mathbf{U}_1 \end{Bmatrix} \mathbf{D}_0^* \begin{Bmatrix} \mathbf{U}_2 \\ \mathbf{U}_2^\dagger \end{Bmatrix}, \quad \begin{pmatrix} LM \\ LE \end{pmatrix} \tag{37}$$

which singles out this elementary operator of the problem from the whole set of bounded matrix operators. In the case of the considered canonical problem of a step discontinuity in a rectangular waveguide the condition Eq. (37) coincides with the property of bilinear scalar product of real-valued eigenfunctions $\operatorname{Im}\left(\boldsymbol{\psi}_p, \boldsymbol{\psi}_q^T\right) = 0$.

Second, the energy conservation law Eq. (29) can be written in the form

$$\left.\begin{matrix} \mathbf{D}_1\mathbf{U}_1 \\ \mathbf{U}_1\mathbf{D}_1^\dagger \end{matrix}\right\} = \mathbf{D}_0\mathbf{U}_2\mathbf{D}_0^\dagger, \quad \left.\begin{matrix} \mathbf{U}_2\mathbf{D}_2^\dagger \\ \mathbf{D}_2\mathbf{U}_2 \end{matrix}\right\} = \mathbf{D}_0^T\mathbf{U}_1\mathbf{D}_0^*. \quad \begin{pmatrix} LM \\ LE \end{pmatrix}. \tag{38}$$

These equalities mean that the numerical range of the operator $\mathbf{D}_p\mathbf{U}_p$ $\left(\mathbf{D}_p\mathbf{U}_p^\dagger\right)$, $p = 1, 2$, lies completely in the fourth (first) quadrant of the complex plane. Based on the well-known geometrical properties of the space ℓ_2 it has been established in paper [20] that on this condition the operator $\mathbf{D}_p$ is an *m*-accretive, i.e., $\operatorname{Re}\mathbf{D}_p > 0$. In this proof the conceptions and ideas of the theory of operators in a space with indefinite metric (in the Pontrjagin space [21] in the case under consideration) are used.

Proceeding from the property of the Cayley transform

$$\operatorname{Re}\mathbf{D}_p > 0 \quad \Leftrightarrow \quad \left\| {}^p\mathbf{R} \right\| < 1 \tag{39}$$

we conclude that the reflection operator represents a strict contraction. Thus, we have **Theorem 2**. The solution of the problem of mode diffraction by a waveguide step discontinuity in the form of the Fresnel formulas for the scattering operators Eq. (20) exists and is unique.

Next, let us introduce into consideration the operator $\mathbf{A}_p \equiv \left(\mathbf{D}_p + \mathbf{I} \right)^{-1} = = \frac{1}{2}\left(\mathbf{I} - \mathbf{R}_p \right)$ for which through direct calculations we find

$$\operatorname{Re}\mathbf{A}_p = \mathbf{A}_p\mathbf{A}_p^\dagger + \mathbf{A}_p\left(\operatorname{Re}\mathbf{D}_p\right)\mathbf{A}_p^\dagger = \mathbf{A}_p^\dagger\mathbf{A}_p + \mathbf{A}_p^\dagger\left(\operatorname{Re}\mathbf{D}_p\right)\mathbf{A}_p > 0. \quad (40)$$

Therefore, this operator represents an *m*-accretive contraction, $\operatorname{Re}\mathbf{A}_p > \mathbf{A}_p\mathbf{A}_p^\dagger$.

Let us define the condition number of this matrix operator after the formula $\operatorname{cond}\left(\mathbf{A}_p\right) \equiv \left\| \mathbf{A}_p \right\| \left\| \mathbf{A}_p^{-1} \right\|$ which yields the following estimate

$$1 \le \operatorname{cond}\left(\mathbf{A}_p\right) \le 1 + \left\| \mathbf{D}_p \right\| < \infty.$$

Ipso facto we have proven the stability of the obtained solution Eq. (20) on the set of bounded operators, which act in the entire space ℓ_2.

8. DIVERSITY OF THE FORMS OF THE PROBLEM SOLUTION

Equivalent transformations of the operator-based Fresnel formulas Eq. (20) lead to other effective forms of the sought-for solution. Let us represent here the found solution through the above introduced operator $\mathbf{A}_p$, $p = 1, 2$, in such tabular format as

$$\begin{matrix} {}^{1}\mathbf{R} = (\pm)\left(\mathbf{I} - 2\mathbf{A}_1\right), & {}^{12}\mathbf{T} = 2\mathbf{A}_1\mathbf{D}_0, \\ {}^{21}\mathbf{T} = 2\mathbf{A}_2\mathbf{D}_0^T, & {}^{2}\mathbf{R} = (\mp)\left(\mathbf{I} - 2\mathbf{A}_2\right). \end{matrix} \quad \begin{pmatrix} LM \\ LE \end{pmatrix}$$

Making use of the generalized scattering matrix **S** this representation can be brought in obvious way to the compact form

$$\mathbf{S} = (\pm)\mathbf{J}_0 + 2\mathbf{A}\left(\mathbf{V}_0 \mp \mathbf{J}_0\right), \quad \begin{pmatrix} LM \\ LE \end{pmatrix}$$

in which the diagonal operator matrices

$$\mathbf{J}_0 = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{bmatrix} \text{ and } \mathbf{V}_0 = \begin{bmatrix} 0 & \mathbf{D}_0 \\ \mathbf{D}_0^T & 0 \end{bmatrix}$$

are, respectively, the canonical symmetry of the space $(\ell_2)^2$, m-accretive contraction and symmetrical operator of the problem. Having extracted the square root of the operator **A** we arrive at the relation

$$\mathbf{A}\left(\mathbf{V}_0 \mp \mathbf{J}_0\right) = \left(\mathbf{V}_0 \mp \mathbf{J}_0\right)^{-1},$$

where the right-hand part of this equality determines the m-accretive operator.

Thus, the sought-for solution takes a simple form, viz.

$$\mathbf{S} = (\pm)\mathbf{J}_0 + 2\left(\mathbf{V}_0 \mp \mathbf{J}_0\right)^{-1} = 2\mathbf{A}^{1/2} \pm \mathbf{J}_0, \begin{pmatrix} LM \\ LE \end{pmatrix}, \tag{41}$$

which is especially convenient for computations.

Another equivalent representation of the solution, which follows from the formula Eq. (41), is the Cayley transform

$$\mathbf{S}\mathbf{J}_0 = \pm\left(\mathbf{J}_0\mathbf{V}_0 \pm \mathbf{J}_0\right)\left(\mathbf{J}_0\mathbf{V}_0 \mp \mathbf{J}_0\right)^{-1}, \begin{pmatrix} LM \\ LE \end{pmatrix} \tag{42}$$

relating the operator matrices

$$\mathbf{S}\mathbf{J}_0 = \begin{bmatrix} {}^{1}\mathbf{R} & -{}^{12}\mathbf{T} \\ {}^{21}\mathbf{T} & -{}^{2}\mathbf{R} \end{bmatrix} \text{ and } \mathbf{J}_0\mathbf{V}_0 = \begin{bmatrix} 0 & \mathbf{D}_0 \\ -\mathbf{D}_0^T & 0 \end{bmatrix}.$$

The expression Eq. (42) can be effectively used for computations as well.

9. CONCLUSIONS

The typical of the standard mode-matching technique formidable problems of proving the existence, uniqueness and robustness of the solution to the infinite system of linear algebraic equations are the corollaries of the generally accepted special formulation of the diffraction problem rather than of the mode-matching technique itself.

The proposed new formulation of the problem of wave diffraction results in natural generalization of the conventional mode-matching technique. In substance, the novel approach implies changing the unknown Fourier coefficients with elements of the sought-for matrix operator of scattering (the matrix operator technique in the theory of diffraction). As a result, the correctness of the obtained matrix model of the generalized mode-matching technique can be proven rigorously.

From this new standpoint the matrix model in the form of an infinite system of linear algebraic equations corresponds by no means to the essence of the mode-matching technique, but rather represents an absolutely unjustified truncation of the general matrix-operator equation which leads to a fatal loss of the necessary information about properties of both the solution itself and the specified operator of the problem.

The above presented fundamentals of the matrix operator technique as applied to the scalar problems of stationary diffraction of waveguide modes make it possible to introduce matrix operators of scattering in a natural manner as the true sought-for unknowns of the mode-matching technique.

The derivation of the Fresnel formulas for the operators of mode reflection and transmission Eq. (20) is demonstrated by way of example of the canonical problem of wave diffraction by a step discontinuity in the *H*- and *E*-plane of a rectangular waveguide.

The obtained results of applying the generalized mode-matching technique can be disseminated on other scalar problems of mode diffraction by an abrupt waveguide discontinuity, i.e., by a discontinuity whose inherent volume is zero. The possibility of separation this class of problems from the variety of the waveguide mode diffraction problems follows from the basic energy laws. Namely, if we write for two-port planar waveguide transformers four basic energy laws for a volume V_d containing a step discontinuity, then by passing to the limit $V_d \to 0$ with account of the edge condition we arrive at the formulas Eqs. (24) and (28). Therefore, the class of mode diffraction problems under consideration is completely defined as such for which the relations Eqs. (25) to (27) and (29) to (31) relating the sought-for matrix operators of scattering hold.

If for each problem of the class in question we assume the existence of a common "operator of the problem" which is determined by the discontinuity geometry and depends on working frequency, then, as has been found, the energy relations Eqs. (25) to (27) imply the existence of the matrix model in the form of the Fresnel formulas for the mode reflection and transmission operators Eq. (20).

It has been proven that the correctness of the operator-based Fresnel formulas is the direct corollary of the complex power conservation law and the second Lorentz lemma written in the operator form Eqs. (29) to (31). Thus, the problem of the rigorous justification of the matrix model of the generalized matching technique has been completely solved.

The derived operator-based Fresnel formulas beget various equivalent forms of the solution which make it possible to disclosure the structure of the generalized scattering matrix. It is advantageous to use the solution in the form of Eqs. (41) and (42) for constructing effective computational algorithms.

The developed and rigorously justified method of analyzing the mode diffraction by waveguide discontinuities can be regarded as a generalization of the conventional mode-matching technique widely used in the applied electrodynamics.

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